

Regularizations of Twisted Covariant Systems and Crossed Products with Continuous Trace

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We define a regularization of a twisted covariant system (G, A, τ) as a G -equivariant continuous map from $\text{Prim}(A)$ into a locally compact G -space Ω with certain additional properties. We show that regularizations can be a useful tool for determining the structure of twisted crossed products, and we use the developed techniques to investigate whether certain twisted crossed products have continuous trace. © 1993 Academic Press, Inc.

1. INTRODUCTION

If a locally compact group G acts strongly continuously on a C^* -algebra A , then the pair (G, A) is called a covariant (or C^* -dynamical) system. In a way similar to the construction of group C^* -algebras of locally compact groups, one can form the crossed product algebra $C^*(G, A)$, and it is well known that the $*$ -representations of $C^*(G, A)$ are in one to one correspondence to the covariant representations of (G, A) . It has become more and more clear in the past thirty years, say, that the representation theory of covariant systems and crossed products plays a very important role in various areas of mathematics. The importance of the theory is even more evident from the introduction of twisted covariant systems and the development of a new and more powerful approach to the Mackey-machine by Green [14] using Rieffel's theory of induced representations of C^* -algebras [25, 26]. Recall that a twisted covariant system is a triple (G, A, τ) consisting of a covariant system (G, A) together with a so-called twisting map τ , where τ is a strictly continuous homomorphism from a closed normal subgroup N_τ of G into the group $\mathcal{U}(A)$ of unitaries in the multiplier algebra $\mathcal{M}(A)$ of A with the properties

$$\tau(xnx^{-1}) = {}^x(\tau(n))$$

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for $x \in G$ and $n \in N_\tau$, and

$$\tau(n) a \tau(n^{-1}) = {}^n a$$

for all $n \in N_\tau$ and $a \in A$. The twisted crossed product $C^*(G, A, \tau)$ is defined as the quotient of $C^*(G, A)$ by the intersection of the kernels of all covariant representations $\pi = (\pi_G, \pi_A)$ of (G, A) which preserve τ ; i.e., which satisfy

$$\pi_G(n) = \pi_A(\tau(n)) \quad \text{for all } n \in N_\tau.$$

The great advantage of twisted covariant systems is the possibility of splitting a given system by a closed normal subgroup, hence making the results for twisted systems completely available in the representation theory of, for example, locally compact groups.

Although Green's results in [14] and the results of Gootman and Rosenberg in [12] give a reasonable understanding of the set of all primitive ideals of a given twisted crossed product $C^*(G, A, \tau)$, there is much less progress in determining the topology of the primitive ideal space, and hence of the global structure of $C^*(G, A, \tau)$. It is the purpose of this paper to develop some techniques which possibly bring more light to this question and to show that these techniques are very useful, at least in some special cases.

We start in Section 2 by introducing regularizations of twisted covariant systems. Roughly speaking these are pairs (Ω, R) where Ω is a locally compact G -space and $R: \text{Prim}(A) \rightarrow \Omega$ a continuous G -equivariant map. The motivation for this is the observation that, in general, to have a good chance of computing the topology of the primitive ideal space of $C^*(G, A, \tau)$ successfully, we need $\text{Prim}(A)$ to have nice properties as a G -space and the stabilizer map $J \rightarrow S_J$ from $\text{Prim}(A)$ into the set $\mathcal{K}(G)$ of all closed subgroups of G equipped with Fell's compact-open topology to be continuous (compare with [4, 11, 22, 24, 32], but see also [31] for the somewhat different case of abelian transformation groups). Unfortunately, $\text{Prim}(A)$ very often is quite ugly as a G -space and sometimes the stabilizer map even has no point of continuity. The main idea of introducing regularizations is to replace the role of $\text{Prim}(A)$ by Ω in the Mackey analysis, hoping that Ω has much better properties as a G -space. This probably leads to a loss of information because the stability groups of the elements of Ω are generally bigger than the stability groups of the primitive ideals and there is no way to use Mackey's little group method in these case. But, nevertheless, there are examples which show that a good regularization can improve the overall situation enormously. This is especially the case when Ω is a locally σ -trivial G -space. Such spaces are

generalizations of principal bundles for the case of non-free actions of non-abelian groups with continuously varying stabilizers and are defined as σ -proper G -spaces, in the sense of Raeburn and Williams [24], which have local continuous sections.

Our main result in Section 3 is a global version of Green's approach to Mackey's theorem [14, Theorem 17] for the case of regularizations (Ω, R) of (G, A, τ) such that Ω is a σ -trivial G -space; i.e., Ω is a σ -proper G -space having a global continuous section. To explain this result let Σ denote a continuous section for the action of G on Ω , and let $S: \Sigma \rightarrow \mathcal{K}(G)$; $\omega \rightarrow S_\omega$ denote the restriction of the stabilizer map to Σ . Since S is assumed to be continuous we can form the subgroup algebra $C^*(\Sigma^S, A, \tau)$ as defined in [4], which can be thought of as a fiber algebra over Ω with fibers $C^*(S_\omega, A, \tau)$. The irreducible representations of this algebra are given by the collection of all pairs (ω, ρ) , where ρ is an irreducible representation of $C^*(S_\omega, A, \tau)$. Now let I_Σ denote the intersection of all kernels of representations $(\omega, \rho) \in C^*(\Sigma^S, A, \tau)^\wedge$ such that $\ker \rho_A \supseteq \ker R^{-1}(\{\omega\})$. Then we see that $C^*(G, A, \tau)$ is Morita-equivalent to $C^*(\Sigma^S, A, \tau)/I_\Sigma$ (Green's theorem is the special of this result where Ω is a homogeneous G -space). If Ω is a locally σ -trivial G -space we get as a consequence that each irreducible representation of $C^*(G, A, \tau)$ has an open neighborhood U which is homeomorphic to the dual space of a quotient $C^*(\Sigma_U^S, A, \tau)/I_{\Sigma_U}$, where Σ_U is an appropriate local section. Hence each element in $C^*(G, A, \tau)^\wedge$ has a neighbourhood which can be described in terms of an appropriate subgroup algebra.

In Section 4 we restrict our attention to σ -trivial regularizations (Ω, R) in which R is assumed to be open and surjective, hence by a result of Lee [15], in which A is isomorphic to the section algebra $\Gamma_0(E)$ of a C^* -bundle $p: E \rightarrow \Omega$ with fibers $A_\omega = A/\ker R^{-1}(\{\omega\})$. Given any C^* -bundle $p: E \rightarrow \Omega$ and a continuous map $H: \Omega \rightarrow \mathcal{K}(G)$, for some locally compact group G , we define an action of $\Omega^H = \{(\omega, x) \in \Omega \times G; x \in H_\omega\}$ on $A = \Gamma_0(E)$ as a family of actions α_ω of H_ω on the fibers A_ω satisfying certain continuity conditions and we give also the notion of twisting maps for such actions. Then, generalizing a construction of subgroup algebras given by Raeburn and Williams in [24], we define the twisted crossed product $\Omega^H \times_{\alpha, \tau} A$ of Ω^H and A . We see that in the case where (Ω, R) is an open σ -trivial regularization the quotient $C^*(\Sigma^S, A, \tau)/I_\Sigma$ is isomorphic to $\Sigma^S \times_{\alpha, \tau} \Gamma_0(E|\Sigma)$, where $E|\Sigma = p^{-1}(\Sigma)$ and the action of Σ^S on $\Gamma_0(E|\Sigma)$ is induced from the action of G on A . If all stability groups are amenable, this algebra is itself a section algebra of a C^* -bundle over Ω with fibers $C^*(S_\omega, A_\omega, \tau_\omega)$. Furthermore, we define induced C^* -algebras $\text{Ind}_{\Omega^H}^G A$ for actions of Ω^H on A , generalizing the well known construction of the induced C^* -algebra derived from an action of a fixed subgroup of G . There is a canonical G -action on $\text{Ind}_{\Omega^H}^G A$, and the main result in Section 4

says that the induced systems are exactly those which have open σ -trivial regularizations.

Our final section, Section 5, deals with the question of whether a given twisted crossed product $C^*(G, A, \tau)$ has continuous trace. Using the results of the previous sections we show that if (Ω, R) is an open locally σ -trivial regularization of the system (G, A, τ) such that S_ω/N_τ is amenable for all $\omega \in \Omega$, then $C^*(G, A, \tau)$ has continuous trace if and only if the twisted crossed product $\Omega^S \times_{\alpha, \tau} A$ has continuous trace. This is an extension of some of Williams' results in [32] about transformation group C^* -algebras with continuous trace. A more sophisticated result can be obtained for the special case where A is of type I and \hat{A} itself is a σ -trivial G -space, and where in addition the action of G on A is locally unitary on the stabilizers relative to τ . The last assumption is somewhat stronger than the assumption that all Mackey obstructions vanish. Covariant systems of abelian groups with those properties have been studied extensively in the literature (see for instance [21, 22, 24]). In the case of varying stabilizers, this was done by Raeburn and Williams in [24], and it was shown that in this case $C^*(G, A)^\wedge$ is a $\hat{\sigma}$ -trivial \hat{G} -space, which implies that $G^*(G, A)^\wedge$ is Hausdorff. However, even in the abelian case it was an open question as to whether $C^*(G, A)$ has continuous trace if A has this property (see [24, Remark 6.11]). The main purpose of Section 5 is filling this gap. In fact we prove that, under the conditions mentioned above, $C^*(G, A, \tau)$ has continuous trace if and only if A and the transformation group algebra $C^*(\hat{G}, \Omega)$ have continuous trace, where $\hat{G} = G/N_\tau$ with the obvious action on $\Omega = \hat{A}$. The answer to the problem mentioned above is then a consequence of Williams' theorem [32, Theorem 4.8], which in this special situation is included in the result explained earlier. But it gives similar answers also in some other interesting cases.

If Ω is a locally compact space and F a normed vector space, we always denote by $C_0(\Omega, F)$ and $C_c(\Omega, F)$ the spaces of continuous F -valued functions which vanish at infinity or which have compact support, respectively. If $F = \mathbb{C}$ we denote them simply by $C_0(\Omega)$ and $C_c(\Omega)$. If A and B are algebras we denote by $A \otimes B$ the algebraic tensor product and by $A \hat{\otimes} B$ the unique C^* -completion if A and B are C^* -algebras and at least one of them is nuclear. Furthermore, we denote by $\mathcal{M}(A)$ the multiplier algebra of the C^* -algebra A and by $\mathcal{U}(A)$ the group of unitaries in $\mathcal{M}(A)$ equipped with the strict topology.

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2. REGULARIZATIONS

Before we start with the definition of regularizations let us first recall some common notions. First of all a subset E of a topological space M is called locally closed in M if E is open in its closure \bar{E} in M . Note that a subset of a locally compact space is locally closed if and only if it is locally compact in the relative topology. Now let A be a C^* -algebra and E a locally closed subset of $\text{Prim}(A)$. A $*$ -representation π of A is said to live on E if

$$\ker \pi \supseteq \ker E \quad \text{but} \quad \ker \pi \not\supseteq \ker(\bar{E} \setminus E),$$

where $\ker E$ denotes the intersection of all elements in E ; i.e., $\ker E = \bigcap_{J \in E} J$.

DEFINITION 1. Let (G, A, τ) be a twisted covariant system and (G, Ω) a locally compact transformation group. Suppose that $R: \text{Prim}(A) \rightarrow \Omega$ is a continuous G -equivariant map and let \mathcal{S}_R be the space of all pairs (ω, ρ) with $\omega \in \Omega$ and $\rho \in C^*(S_\omega, A, \tau)^\wedge$ such that $\ker \rho_A \supseteq \ker R^{-1}(\{\omega\})$.

(1) The pair (Ω, R) is called a *regularization* of (G, A, τ) if the map

$$\text{Ind}: \mathcal{S}_R \rightarrow \text{Prim}(C^*(G, A, \tau)); (\omega, \rho) \rightarrow \ker(\text{ind}_{S_\omega}^G \rho)$$

is well defined and surjective.

(2) (Ω, R) is called a *complete regularization* if for every $\omega \in \Omega$ the canonical map $G/S_\omega \rightarrow G(\omega)$; $xS_\omega \rightarrow x\omega$ is a homeomorphism and, in addition, if for all $\pi \in C^*(G, A, \tau)^\wedge$ there exists an $\omega \in \Omega$ such that π_A lives on $R^{-1}(G(\omega))$.

As a first step we investigate under which conditions a given map $R: \text{Prim}(A) \rightarrow \Omega$ is a regularization. The following result turns out to be an easy consequence of a theorem of Gootman and Rosenberg [12, Theorem 4.2].

PROPOSITION 1. Suppose (G, A, τ) is a twisted covariant system such that G and A are separable and G/N_τ is amenable. Then every G -equivariant continuous map $R: \text{Prim}(A) \rightarrow \Omega$ is a regularization.

Proof. Let $\omega \in \Omega$ and $J \in R^{-1}(\{\omega\})$. Then it is easily seen that the stabilizer S_ω of ω contains the stabilizer S_J of J . Now let $\rho \in C^*(S_\omega, A, \tau)^\wedge$ such that $\ker \rho_A \supseteq \ker R^{-1}(\{\omega\})$. Then by [12, Theorem 4.2] there exist $J \in R^{-1}(\{\omega\})$ and $\sigma \in C^*(S_J, A, \tau)^\wedge$ such that $\ker \sigma_A = J$ and $\ker(\text{ind}_{S_J}^G \sigma) = \ker \rho$ in $C^*(S_\omega, A, \tau)$. Again by [12, Theorem 4.2] it follows that $\ker(\text{ind}_{S_\omega}^G \rho) = \ker(\text{ind}_{S_J}^G \sigma)$ is a primitive ideal in $C^*(G, A, \tau)$. Hence the map $\text{Ind}: \mathcal{S}_R \rightarrow \text{Prim}(G, A, \tau)$ is well defined.

To see that this map is onto, let I be a primitive ideal in $C^*(G, A, \tau)$. Then [12, Theorem 4.2] implies the existence of some $J \in \text{Prim}(A)$ and a $\sigma \in C^*(S_J, A, \tau)^\wedge$ such that $\ker \sigma_A = J$ and $\ker(\text{ind}_{S_J}^G \sigma) = I$. Let $\omega = R(J)$. Thus $\text{ind}_{S_J}^{S_\omega} \sigma$ is a primitive ideal in $C^*(S_\omega, A, \tau)$ and there exists a $\rho \in C^*(S_\omega, A, \tau)^\wedge$ such that $\ker \rho = \ker(\text{ind}_{S_J}^{S_\omega} \sigma)$. But this implies that $I = \ker(\text{ind}_{S_\omega}^G \rho)$. Furthermore, since $\ker \rho_A = \bigcap_{x \in S_\omega} {}^x J \supseteq \ker R^{-1}(\{\omega\})$, we see that $\rho \in \mathcal{S}_R$, which finishes the proof. ■

In our next proposition we use Green's theorem [14, Theorem 17] to show that the notion of complete regularizations is indeed stronger than the notion of regularizations.

PROPOSITION 2. *Suppose that (Ω, R) is a complete regularization of (G, A, τ) . Then $\text{ind}_{S_\omega}^G \rho$ is irreducible for all $\rho \in \mathcal{S}_R$ and the map*

$$\text{ind}: \mathcal{S}_R \rightarrow C^*(G, A, \tau)^\wedge; (\omega, \rho) \rightarrow \text{ind}_{S_\omega}^G \rho$$

is surjective. In particular, (Ω, R) is a regularization of (G, A, τ) . Furthermore, if $(\omega, \rho), (\omega', \rho') \in \mathcal{S}_R$, then $\text{ind}_{S_\omega}^G \rho = \text{ind}_{S_{\omega'}}^G \rho'$ if and only if there exists an $x \in G$ such that $\omega = x\omega'$ and $\rho = {}^x \rho'$, where ${}^x \rho'$ denotes the conjugate of ρ' by x .

Proof. Let $\omega \in \Omega$. Since the G -orbit $G(\omega)$ is homeomorphic to G/S_ω it is in particular locally closed. Hence $R^{-1}(G(\omega))$ is locally closed, too. Thus we can find G -invariant closed ideals $I_1 \subseteq I_2$ in A such that $R^{-1}(G(\omega))$ is homeomorphic to $\text{Prim}(I_2/I_1)$. Furthermore, if $K = \ker R^{-1}(\{\omega\})$ then $R^{-1}(\{\omega\})$ is canonically homeomorphic to $\text{Prim}(I_2/I_2 \cap K)$. Now let $\mathcal{Q}_\omega = \{\pi \in C^*(G, A, \tau)^\wedge; \pi_A \text{ lives on } R^{-1}(G(\omega))\}$, $\mathcal{S}_\omega = \{\rho \in C^*(S_\omega, A, \tau)^\wedge; \ker \rho_A \supseteq K\}$, and let τ' and τ'' denote the twisting maps on N_τ with images in $\mathcal{M}(I_2/I_1)$ and $\mathcal{M}(I_2/I_2 \cap K)$ defined by $\tau'(n) = \tau(n) + I_1$ and $\tau''(n) = \tau(n) + (I_2 \cap K)$, respectively. Then \mathcal{Q}_ω and \mathcal{S}_ω can be canonically identified with $C^*(G, I_2/I_1, \tau')^\wedge$ and $C^*(S_\omega, I_2/I_2 \cap K, \tau'')^\wedge$, respectively. By [14, Theorem 17], $C^*(S_\omega, I_2/I_2 \cap K, \tau'')$ is Morita-equivalent to $C^*(G, I_2/I_1, \tau')$ which implies that the inducing map restricted to \mathcal{S}_ω establishes a homeomorphism between \mathcal{S}_ω and \mathcal{Q}_ω . Since $C^*(G, A, \tau)^\wedge = \bigcup_{\omega \in \Omega} \mathcal{Q}_\omega$ and $\mathcal{S}_R = \bigcup_{\omega \in \Omega} \mathcal{S}_\omega$ it follows that ind is in fact a surjective map from \mathcal{S}_R onto $C^*(G, A, \tau)^\wedge$.

Now suppose that $(\omega, \rho), (\omega', \rho') \in \mathcal{S}_R$ such that $\text{ind}_{S_\omega}^G \rho = \text{ind}_{S_{\omega'}}^G \rho'$. Then it is easily seen that ω and ω' are contained in the same G -orbit; i.e., there exists an $x \in G$ such that $\omega = x\omega'$. But this implies that (ω, ρ) and $(\omega, {}^x \rho')$ are both elements of \mathcal{S}_ω which have the same image in \mathcal{Q}_ω . Hence $\rho = {}^x \rho'$. ■

Recall now that a topological space M is called almost Hausdorff if every closed subset E of M contains a relatively open dense Hausdorff subset.

The following proposition gives some conditions on (G, Ω) which imply that any continuous G -equivariant map $R: \text{Prim}(A) \rightarrow \Omega$ is a complete regularization. The proof uses some ideas given in the proof of [26, Proposition 8.1].

PROPOSITION 3. *Let (G, A, τ) be a twisted covariant system, (G, Ω) a locally compact transformation group, and $R: \text{Prim}(A) \rightarrow \Omega$ a continuous G -equivariant map. Then (Ω, R) is a complete regularization of (G, A, τ) if it satisfies one of the following conditions:*

- (1) *The canonical map $G/S_\omega \rightarrow G(\omega)$; $xS_\omega \rightarrow x\omega$ is a homeomorphism for each $\omega \in \Omega$, and the orbit space Ω/G is almost Hausdorff or second countable.*
- (2) *Ω/G is almost Hausdorff and G is σ -compact.*
- (3) *G and Ω are second countable and Ω/G is a T_0 -space.*

Proof. It was shown in [26, Proposition 7.1] that Condition 1 is a consequence of Condition 2, and it is a classical result of Glimm [11, Theorem 1] that Condition 3 also implies Condition 1. Hence it is enough to show that Condition 1 implies that R is a complete regularization.

So let $\pi \in C^*(G, A, \tau)^\wedge$. We have to show that there exists an $\omega \in \Omega$ such that π_A lives on $R^{-1}(G(\omega))$. Let us denote by \mathcal{F} the set of all G -invariant closed subsets F of Ω such that π_A lives on $R^{-1}(F)$. Then $\mathcal{F} \neq \emptyset$ since $\Omega \in \mathcal{F}$. If $D = \bigcap_{F \in \mathcal{F}} F$, then D is also a G -invariant closed subset of Ω , and since $\ker \pi_A \subseteq \ker R^{-1}(F)$ for all $F \in \mathcal{F}$ the same is true for D . Hence $D \in \mathcal{F}$.

Now let C_1 and C_2 be closed G -invariant subsets of D such that $C_1 \cup C_2 = D$. We show that $C_i = D$ for at least one $i \in \{1, 2\}$. For this let $I_i = \ker R^{-1}(C_i)$, $i = 1, 2$. Then $I_1 \cdot I_2 = \ker R^{-1}(D)$. Assume now that C_1 and C_2 are proper subsets of D . Then it follows that $\pi_A(I_i) \neq \{0\}$ for $i = 1, 2$, since the equality would imply that π_A lives on C_i , which contradicts the construction of D . Hence $\pi_A(I_i) \mathcal{H}_\pi$ is dense in \mathcal{H}_π for $i = 1, 2$. But this implies that $\pi_A(\ker R^{-1}(D)) \mathcal{H}_\pi = \pi_A(I_1 \cdot I_2) \mathcal{H}_\pi$ is also dense in \mathcal{H}_π , which contradicts the fact that π_A lives on $R^{-1}(D)$. Since Ω/G is almost Hausdorff or separable, it follows from the lemma on page 222 in [14] that D is the closure of a G -orbit, say $G(\omega)$.

We now conclude by showing that π_A lives on $R^{-1}(G(\omega))$. First we observe that π_A does not live on $R^{-1}(\overline{G(\omega)} \setminus G(\omega))$ since this would contradict the construction of D . Hence π_A also does not live on $\overline{R^{-1}(\overline{G(\omega)})} \setminus R^{-1}(G(\omega))$ since this is a subset of $R^{-1}(\overline{G(\omega)} \setminus G(\omega))$. The fact that $\ker \pi_A \supseteq \ker R^{-1}(G(\omega))$ follows now from the facts that $\ker \pi_A \supseteq \ker R^{-1}(\overline{G(\omega)}) = \ker R^{-1}(G(\omega)) \cdot \ker R^{-1}(\overline{G(\omega)} \setminus G(\omega))$ and $\pi_A(\ker R^{-1}(\overline{G(\omega)} \setminus G(\omega))) \mathcal{H}_\pi$ is dense in \mathcal{H}_π . ■

Before we proceed to investigate the properties of regularizations let us first give a canonical example where regularizations naturally occur.

EXAMPLE 1. Let (G, A) be a covariant system and (G, Ω) a locally compact transformation group. Then there is a canonical action of G on the algebra $C_0(\Omega, A)$ of all A -valued continuous functions on Ω which vanish at infinity. This action is defined by

$${}^x\varphi(\omega) = {}^x(\varphi(x^{-1}\omega)), \quad \varphi \in C_0(\Omega, A), \quad x \in G, \quad \omega \in \Omega,$$

and is usually called the diagonal action of G on $C_0(\Omega, A)$.

The projection

$$R: \text{Prim}(C_0(\Omega, A)) \rightarrow \Omega; (\omega, J) \rightarrow \omega$$

is always continuous and G -equivariant. Hence, if G , A , and Ω are separable and G is amenable, then (Ω, R) is a regularization for $C^*(G, C_0(\Omega, A))$. Furthermore, if (G, Ω) satisfies one of the conditions in Proposition 3 then (Ω, R) is a complete regularization.

In this example the set \mathcal{S}_R can be identified with the disjoint union $\bigcup_{\omega \in \Omega} (S_\omega, A)^\wedge$. In order to prove this let $\omega \in \Omega$. Then

$$\ker R^{-1}(\{\omega\}) = I_\omega = \{\varphi \in C_0(\Omega, A); \varphi(\omega) = 0\}.$$

Hence, for $\rho \in C^*(S_\omega, C_0(\Omega, A))^\wedge$ it follows that $\rho \in \mathcal{S}_R$ if and only if ρ is a representation of $C^*(S_\omega, C_0(\Omega, A)/I_\omega)$. But $C_0(\Omega, A)/I_\omega$ is isomorphic to A by the map $\varphi + I_\omega \rightarrow \varphi(\omega)$. The map R in this example has the additional property that it is open and surjective. We investigate regularizations with this property more deeply in Section 3.

In the remaining parts of this paper we always assume that the stabilizer map $S: \Omega \rightarrow \mathcal{K}(G)$ is continuous, where $\mathcal{K}(G)$ denotes the space of all closed subgroups of G equipped with Fell's compact-open topology as defined in [6]. In this case we can imbed \mathcal{S}_R canonically into the dual space of a so-called subgroup algebra. Let us first recall the basic definitions of these algebras as given in [4]. To this end let Ω be any locally compact space, (G, A) a covariant system and $H: \Omega \rightarrow \mathcal{K}(G)$ a continuous map. Then

$$\Omega^H = \{(\omega, x) \in \Omega \times G; x \in H_\omega\}$$

is a locally compact Hausdorff space. If (G, A) is a covariant system, then the Ω^H -subgroup algebra $C^*(\Omega^H, A)$ is defined as the C^* -completion of the algebra $C_c(\Omega^H, A)$ consisting of all continuous A -valued functions with

compact support, where multiplication and involution on $C_c(\Omega^H, A)$ are given by

$$f * g(\omega, x) = \int_{H_\omega} f(\omega, y) {}^x(g(\omega, y^{-1}x)) d_{H_\omega} y$$

and

$$f^*(\omega, x) = \Delta_{H_\omega}(x^{-1}) {}^x(f(\omega, x^{-1})^*).$$

Here $(d_H)_{H \in \mathcal{K}(G)}$ denotes a smooth choice of Haar measures on $\mathcal{K}(G)$ and Δ_{H_ω} the modular function on H_ω . $C^*(\Omega^H, A)$ can be thought of as a fiber-algebra over the base space Ω with fibers $C^*(H_\omega, A)$, and we see later that, at least in the case that G is amenable, $C^*(\Omega^H, A)$ is in fact isomorphic to a section algebra of a C^* -bundle with base space Ω . The irreducible representations of $C^*(\Omega^H, A)$ can be identified with the collection of all pairs (ω, ρ) , $\omega \in \Omega$, $\rho \in C^*(H_\omega, A, \tau)^\wedge$ by $(\omega, \rho)(f) = \rho(f(\omega, \cdot))$.

If (G, A, τ) is a twisted covariant system and $H_\omega \supseteq N_\tau$ for all $\omega \in \Omega$, then the Ω^H -subgroup algebra of (G, A, τ) is defined as the quotient of $C^*(\Omega^H, A)$ by the intersection of the kernels of all irreducible representations of $C^*(\Omega^H, A)$ which preserve τ .

Now suppose that (Ω, R) is a regularization of the twisted covariant system (G, A, τ) such that the stabilizer map $S: \Omega \rightarrow \mathcal{K}(G)$; $\omega \rightarrow S_\omega$ is continuous. Then it is clear that \mathcal{S}_R can be naturally imbedded into $C^*(\Omega^S, A, \tau)^\wedge$. The next theorem is a direct consequence of [4, Theorem 4]. Recall that for any topological G -space M the quasi-orbit space $\mathcal{Q}(M)$ is the quotient space of M by the equivalence relation $m \sim m' \Leftrightarrow \overline{G(m)} = \overline{G(m')}$.

THEOREM 1. *Let (Ω, R) be a regularization of (G, A, τ) such that the stabilizer map $S: \Omega \rightarrow \mathcal{K}(G)$ is continuous. Then*

$$\text{Ind}: \mathcal{Q}(\mathcal{S}_R) \rightarrow \text{Prim}(G, A, \tau); [(\omega, \rho)] \rightarrow \ker(\text{ind}_{S_\omega}^G \rho)$$

is a homeomorphism, where the action of $x \in G$ on $(\omega, \rho) \in \mathcal{S}_R$ is given by ${}^x(\omega, \rho) = (x\omega, {}^x\rho)$.

This theorem gives a first hint that regularizations might be useful for the description of the topology of the primitive ideal spaces of twisted covariant systems. Note that in the case of a continuous stabilizer map $S': \text{Prim}(A) \rightarrow \mathcal{K}(G)$ this map itself may be viewed as a regularization of (G, A, τ) . In this case there is a result similar to the theorem above [4, Theorem 3], in which the representations are actually induced from the stabilizers of the primitive ideals in A . But there are a lot of examples in which the stabilizer map on $\text{Prim}(A)$ is not continuous, but in which there

exist useful regularizations with continuous stabilizer maps. The last result in this section shows that \mathcal{S}_R is always closed in $C^*(\Omega^S, A, \tau)^\wedge$, hence that \mathcal{S}_R is the dual space of a quotient of the subgroup algebra.

PROPOSITION 4. *The set \mathcal{S}_R is closed in $C^*(\Omega^S, A, \tau)^\wedge$.*

Proof. Let $((\omega_i, \rho_i))_{i \in I}$ be a net in \mathcal{S}_R converging to some $(\omega, \rho) \in C^*(\Omega^S, A, \tau)$. Since the restriction of representations is continuous [4, Corollary 2], it follows that $\rho_{A_i} \rightarrow \rho_A$ in $\text{Rep}(A)$. Let $J \in \text{Prim}(A)$ such that $J \supseteq \ker \rho_A$. Then, by passing to a subnet if necessary, we can find $J_i \in \text{Prim}(A)$ with $J_i \supseteq \ker \rho_{A_i}$ for all $i \in I$ such that $J_i \rightarrow J$ in $\text{Prim}(A)$. But $J_i \in R^{-1}(\{\omega_i\})$ for all $i \in I$. Hence, by the continuity of R we observe that $J \in R^{-1}(\{\omega\})$. Since this is true for every primitive ideal containing $\ker \rho_A$ it follows that $\ker \rho_A \supseteq \ker R^{-1}(\{\omega\})$. Thus $(\omega, \rho) \in \mathcal{S}_R$. ■

3. σ -TRIVIAL REGULARIZATIONS

In this section we investigate regularizations (Ω, R) of systems (G, A, τ) where G acts very nicely on Ω . To make this more precise let Ω be a locally compact space, G a locally compact group, and $H: \Omega \rightarrow \mathcal{K}(G)$ a continuous map. We define an equivalence relation \sim^H on $\Omega \times G$ by

$$(\omega, x) \sim^H (\omega', x') \Leftrightarrow \omega = \omega' \text{ and } x \in x'H_{\omega}.$$

Let us denote by $\Omega \times_H G$ the quotient space $(\Omega \times G)/\sim^H$. It follows from [32, Lemma 2.12] that $\Omega \times_H G$ is a locally compact Hausdorff space and that the quotient map from $\Omega \times G$ onto $\Omega \times_H G$ is open. $\Omega \times_H G$ can be viewed as a fibration over Ω with fibers G/H_{ω} . We denote elements of $\Omega \times_H G$ always by pairs (ω, \dot{x}) , where \dot{x} denotes the coset space xH_{ω} .

Now let (G, Ω) be a locally compact transformation group such that the stabilizer map $S: \Omega \rightarrow \mathcal{K}(G)$ is continuous. Then the action of G on Ω is called σ -proper if the map

$$\Omega \times_S G \rightarrow \Omega \times \Omega, \quad (\omega, \dot{x}) \rightarrow (\omega, x\omega)$$

is proper in the usual sense. σ -proper actions were introduced in [24, Definition 4.1]. It was shown that Ω/G is always Hausdorff and that the canonical map $G/S_{\omega} \rightarrow G(\omega)$; $xS_{\omega} \rightarrow x\omega$ is a homeomorphism for each $\omega \in \Omega$ if G acts σ -properly on Ω . We now define σ -trivial and locally σ -trivial G -spaces and regularizations.

DEFINITION 2. Let (G, Ω) be a locally compact transformation group such that G acts σ -properly on Ω .

(1) Ω is called a σ -trivial G -space if there exists a continuous section for Ω/G ; i.e., a closed subset $\Sigma \subseteq \Omega$ such that the quotient map $q: \Omega \rightarrow \Omega/G$ restricted to Σ is a homeomorphism onto Ω/G .

(2) Ω is called a locally σ -trivial G -space if for each $\omega \in \Omega$ there exists a G -invariant neighborhood U such that U is a σ -trivial G -space.

Furthermore, if (G, A, τ) is a twisted covariant system and $R: \text{Prim}(A) \rightarrow \Omega$ is a G -equivariant continuous map, then (Ω, R) is called a σ -trivial or locally σ -trivial regularization if Ω is a σ -trivial or locally σ -trivial G -space, respectively.

It follows from [24, Proposition 4.3] that our notion of σ -trivial and locally σ -trivial G -spaces extends the definition of such spaces given in [24] for abelian groups to arbitrary locally compact groups. The following analogue of [24, Proposition 4.3] for arbitrary groups follows from the same arguments as given by Raeburn and Williams in [24].

PROPOSITION 5. *Let $H: \Sigma \rightarrow \mathcal{K}(G)$ be a continuous map. If we define an action of G on $\Sigma \times_H G$ by $x(\omega, y) = (\omega, xy)$, then $\Sigma \times_H G$ becomes a σ -trivial G -space with section $\Sigma = \{(\omega, e); \omega \in \Sigma\}$. Conversely, if Ω is any σ -trivial G -space with section Σ , then Ω is G -homeomorphic to $\Sigma \times_S G$, where S is the stabilizer map restricted to Σ .*

The next theorem is the main result of this section.

THEOREM 2. *Let (Ω, R) be a σ -trivial regularization of the twisted covariant system (G, A, τ) and let Σ be a continuous section for Ω/G . Furthermore, let*

$$I_\Sigma = \bigcap \{ \ker(\omega, \rho); (\omega, \rho) \in \mathcal{S}_R \text{ and } \omega \in \Sigma \} \subseteq C^*(\Sigma^S, A, \tau).$$

Then $C^(G, A, \tau)$ is Morita-equivalent to $C^*(\Sigma^S, A, \tau)/I_\Sigma$.*

The main tool for the proof of this theorem is the imprimitivity theorem for subgroup algebras [4, Theorem 1]. We start the proof with the following lemma.

LEMMA 1. *Suppose that (G, A) is a covariant system, Ω a locally compact space, and $H: \Omega \rightarrow \mathcal{K}(G)$ a continuous map. Then $C^*(\Omega^H, A)$ is Morita-equivalent to $C^*(G, C_0(\Omega \times_H G, A))$, where G acts on $C_0(\Omega \times_H G, A)$ by the diagonal action.*

Proof. Let Ω_H^G be the quotient space of $\Omega \times G \times G$ by the equivalence relation

$$(\omega, x, y) \sim (\omega', x', y') \Leftrightarrow \omega = \omega', \quad x = x', \quad \text{and} \quad y \in y'H_{\omega'}.$$

Then Ω_H^G is homeomorphic to $G \times (\Omega \times_H G)$ by the map $(\omega, x, \dot{y}) \rightarrow (x, (\omega, \dot{y}))$. If we define multiplication and involution on $C_c(\Omega_H^G, A)$ by

$$F * F'(\omega, x, \dot{y}) = \int_G F(\omega, z, \dot{y}) \cdot (F'(\omega, z^{-1}x, \dot{z}^{-1}\dot{y})) dz$$

and

$$F^*(\omega, x, \dot{y}) = \Delta_G(x^{-1})^{-1} (F(\omega, x^{-1}, \dot{x}^{-1}\dot{y})^*),$$

$F, F' \in C_c(\Omega_H^G, A)$, then it is a special case of [4, Corollary 1] that $C^*(\Omega_H^G, A)$ is Morita-equivalent to the C^* -completion $C^*(\Omega_H^G, A)$ of $C_c(\Omega_H^G, A)$. We claim that the homeomorphism between Ω_H^G and $G \times (\Omega \times_H G)$ induces an isomorphism between $C^*(\Omega_H^G, A)$ and $C^*(G, C_0(\Omega \times_H G, A))$. For this recall that $C_c(G \times (\Omega \times_H G), A)$ can be viewed canonically as a dense subalgebra of $C^*(G, C_0(\Omega \times_H G, A))$. Now we define $\Phi: C_c(\Omega_H^G, A) \rightarrow C_c(G \times (\Omega \times_H G), A)$ by $\Phi F(x, (\omega, \dot{y})) = F(\omega, x, \dot{y})$. Then easy computations show that Φ preserves multiplication and involution. Hence the claim follows from the corollary in [14, p. 203] and the fact that Φ is an isomorphism with respect to the inductive limit topologies. ■

Before we proceed with the proof of the theorem we have to say something about a certain realization of induced representations. For this let (G, A) be a covariant system, H a closed subgroup of G , and $\rho = (\rho_H, \rho_A)$ a covariant representation of (H, A) . Let $\mathcal{H}_{\text{ind}_H^G \rho}$ denote Blattner's construction of the Hilbert space of the induced representation $\text{ind}_H^G \rho_H$ of G [1]. Then the induced representation $\text{ind}_H^G \rho$ of (G, A) can be realized on $\mathcal{H}_{\text{ind}_H^G \rho}$ by the covariant pair $(\text{ind}_H^G \rho_H, \text{ind}_H^G \rho_A)$, where $\text{ind}_H^G \rho_A$ is defined by

$$(\text{ind}_H^G \rho_A(a) \xi)(x) = \rho_A(x^{-1}a)(\xi(x)),$$

$a \in A$, $x \in G$, and $\xi \in \mathcal{H}_{\text{ind}_H^G \rho}$. Furthermore, if P^ρ is the representation of $C_0(G/H)$ on $\mathcal{H}_{\text{ind}_H^G \rho}$ given by

$$(P^\rho(\varphi) \xi)(x) = \varphi(\dot{x})(\xi(x)),$$

$x \in G$, $\varphi \in C_0(G/H)$, and $\xi \in \mathcal{H}_{\text{ind}_H^G \rho}$, then $(\text{ind}_H^G \rho_H, P^\rho \otimes \text{ind}_H^G \rho_A)$ is a covariant pair for $(G, C_0(G/H, A))$ and its integrated form is equivalent to the representation of $G^*(G, C_0(G/H, A))$ which is induced from ρ via the $C_c(H, A) - C_c(G, C_0(G/H, A))$ imprimitivity bimodule $C_c(G, A)$. The intertwining operator is given in [4, Section 5].

We are now ready to proof the theorem.

Proof of Theorem 2. We assume first that τ is trivial. It follows from Lemma 1 that $C^*(\Sigma^S, A)$ is Morita-equivalent to $C^*(G, C_0(\Sigma \times_S G, A))$,

and it is a consequence of Proposition 5 that this algebra is isomorphic to $C^*(G, C_0(\Omega, A))$. For $(\omega, \rho) \in C^*(\Sigma^S, A)$ let $V^{(\omega, \rho)}$ denote the representation of $C^*(G, C_0(\Omega, A))$ induced from (ω, ρ) via the $C^*(\Sigma^S, A) - C^*(G, C_0(\Omega, A))$ imprimitivity bimodule (see [25] for the definitions). We define

$$I_{V(\Sigma)} = \bigcap \{ \ker V^{(\omega, \rho)}; (\omega, \rho) \in \mathcal{S}_\Sigma \},$$

where \mathcal{S}_Σ denotes the set of all $(\omega, \rho) \in \mathcal{S}_R$ such that $\omega \in \Sigma$. Then [26, Proposition 3.2] shows that $C^*(\Sigma^S, A)/I_\Sigma$ is Morita-equivalent to $C^*(G, C_0(\Omega, A))/I_{V(\Sigma)}$.

We claim that $C^*(G, C_0(\Omega, A))/I_{V(\Sigma)}$ is isomorphic to $C^*(G, A)$. By [4, Lemma 8] there is a surjective $*$ -homomorphism Ψ from $C^*(G, C_0(\Omega, A))$ onto $C^*(G, A)$. Hence Ψ defines an injective map Ψ^* from $C^*(G, A)^\wedge$ into $C^*(G, C_0(\Omega, A))^\wedge$ given by $\Psi^*(\pi) = \pi \circ \Psi$, and we get the following diagram of maps.

$$\begin{array}{ccc} C^*(G, C_0(\Omega, A))^\wedge & \xleftarrow{\Psi^*} & C^*(G, A)^\wedge \\ & \swarrow \text{ind} \quad \searrow V & \\ & \mathcal{S}_\Sigma & \end{array}$$

Since by Proposition 2 $\text{ind}: \mathcal{S}_\Sigma \rightarrow C^*(G, A)^\wedge$ is a bijection, the claim follows if we can show that this diagram commutes. In order to show this we have to look for the construction of the representation $V^{(\omega, \rho)}$ for $(\omega, \rho) \in C^*(\Sigma^S, A)^\wedge$. For this let $U^{(\omega, \rho)}$ be the representation of $C^*(\Sigma_S^G, A)$ induced from (ω, ρ) via the imprimitivity bimodule $C_c(\Sigma \times G, A)$ (see [4] for more details). Then [4, Proposition 5] shows that $U^{(\omega, \rho)}$ is equivalent to the pair (ω, U^ρ) , where U^ρ denotes the representation of $C^*(G, C_0(G/S_\omega, A))$ on $\mathcal{H}_{\text{ind}_{S_\omega}^G \rho}$ given by the covariant pair $(\text{ind}_{S_\omega}^G \rho_{S_\omega}, P^\rho \otimes \text{ind}_{S_\omega}^G \rho_A)$. Now we get for all $F \in C_c(\Sigma_S^G, A)$, $\xi \in \mathcal{H}_{\text{ind}_{S_\omega}^G \rho}$, and $x \in G$:

$$\begin{aligned} (U^{(\omega, \rho)}(F) \xi)(x) &= (U^\rho(F(\omega, \cdot, \cdot)) \xi)(x) \\ &= \int_G ((P^\rho \otimes \text{ind}_{S_\omega}^G \rho_A)(F(\omega, y, \cdot)) \text{ind}_{S_\omega}^G \rho_{S_\omega}(y) \xi)(x) dy \\ &= \int_G \rho_A(x^{-1}(F(\omega, y, \dot{x}))) \xi(y^{-1}x) dy. \end{aligned}$$

The isomorphism Φ from $C^*(G, C_0(\Omega, A))$ onto $C^*(\Sigma_S^G, A)$ is given on the dense subalgebra $C_c(G \times \Omega, A)$ by $\Phi f(\omega, y, \dot{x}) = f(y, x\omega)$. Hence for all $f \in C_c(G \times \Omega, A)$ we get

$$\begin{aligned}
 (V^{(\omega, \rho)}(f) \xi)(x) &= (U^{(\omega, \rho)}(\Phi f) \xi)(x) \\
 &= \int_G \rho_A(x^{-1}(f(y, x\omega))) \xi(y^{-1}x) dy.
 \end{aligned}$$

But this expression is equal to $(\text{ind}_{S_\omega}^G \rho(\Psi f) \xi)(x)$ by [4, Lemma 9] which proves the claim and the theorem in the case of trivial τ . The case of non-trivial τ follows easily from [26, Proposition 3.2] and the fact that induced representations preserve τ if and only if the original ones do the same. ■

We finish this section with some corollaries. The first follows immediately from Lemma 1 and Proposition 5.

COROLLARY 1. *Let (G, A) be a covariant system and Ω a σ -trivial G -space with section Σ . Then $C^*(G, C_0(\Omega, A))$ is Morita-equivalent to $C^*(\Sigma^S, A)$.*

For the next corollary recall the fact that in any C^* -algebra A there exists a one to one correspondence between the closed two-sided ideals in A and the open subsets of \hat{A} , which is given by $I \rightarrow \hat{I} \subseteq \hat{A}$ [2]. Furthermore, if (G, A, τ) is a twisted covariant system and J is a G -invariant ideal in A , then there exists an exact sequence

$$0 \rightarrow C^*(G, J, \tau_J) \rightarrow C^*(G, A, \tau) \rightarrow C^*(G, A/J, \tau_{A/J}) \rightarrow 0,$$

where the actions of G on J and A/J are induced from the action of G on A , and where the twisting maps τ_J and $\tau_{A/J}$ are given by

$$\tau_J(n)b = \tau(n)b \quad \text{and} \quad \tau_{A/J}(n)(a+J) = \tau(n)a + J,$$

$n \in N_\tau$, $b \in J$, and $a \in A$.

COROLLARY 2. *Let (Ω, R) be a locally σ -trivial regularization of the twisted covariant system (G, A, τ) . Then for each $\pi \in C^*(G, A, \tau)^\wedge$ there exists a G -invariant ideal J in A , and a section Σ for $R(\text{Prim}(J))/G$, such that $\pi \in C^*(G, J, \tau_J)^\wedge$ and $C^*(G, J, \tau_J)$ is Morita-equivalent to $C^*(\Sigma^S, J, \tau_J)/I_\Sigma$, where*

$$I_\Sigma = \bigcap \{ \ker(\omega, \rho); (\omega, \rho) \in \mathcal{S}_R, \omega \in \Sigma \} \subseteq C^*(\Sigma^S, J, \tau_J).$$

Proof. Let $\pi \in C^*(\Omega, A, \tau)^\wedge$. Then by Proposition 2 there exists an $(\omega, \rho) \in \mathcal{S}_R$ such that $\pi = \text{ind}_{S_\omega}^G \rho$. Since (Ω, R) is a locally σ -trivial regularization we can find a G -invariant open neighborhood $U \subseteq \Omega$ and a closed subset Σ of U (in the relative topology) such that Σ is a continuous section for U/G . Now let $J = \ker R^{-1}(\Omega \setminus U)$. Since $\pi = \text{ind}_{S_\omega}^G \rho$ for some

$(\omega, \rho) \in \mathcal{S}_R$ such that $\omega \in U$ we have $\pi \in C^*(G, J, \tau_J)^\wedge$ and the proof follows from Theorem 2. ■

Remark 1. One could ask whether a result like Theorem 2 can be true also for non- σ -trivial actions on Ω , replacing the section Σ by the orbit space Ω/G . Generally, there would be great difficulty in replacing the quotient $C^*(\Sigma^S, A, \tau)/I_\Sigma$ by an appropriate object, since the stabilizer map is in general not constant on G -orbits. However, if $A = C_0(\Omega)$ and G acts freely and properly on Ω there is a well known result of Green (see [13] and [28]) which says that $C^*(G, C_0(\Omega))$ is Morita-equivalent to $C_0(\Omega/G)$. Hence one could ask if, at least in the case of σ -proper actions of abelian groups on Ω , $C^*(G, \Omega)$ is Morita-equivalent to $C^*((\Omega/G)^{\tilde{S}})$, where $\tilde{S}: \Omega/G \rightarrow \mathcal{K}(G)$ is induced from the stabilizer map which in this case is constant on G -orbits. Unfortunately, this is not true in general since by [22, Example 4.6] there exists an action of \mathbb{R} on S^3 such that $C^*(\mathbb{R}, C_0(S^3))$ has continuous trace with non-zero Dixmier–Douady class, but $C^*((S^3/\mathbb{R})^{\tilde{S}})$ is commutative and therefore has trivial Dixmier–Douady class.

4. INDUCED SYSTEMS AND OPEN REGULARIZATIONS

In this section we investigate open regularizations; i.e., regularizations (Ω, R) in which the map R is open and surjective. Note that we have not assumed any of these conditions before. Let us first recall some basic facts about C^* -bundles and section algebras of C^* -bundles.

A C^* -bundle $p: E \rightarrow \Omega$ is a Banach-bundle in the sense of Fell (see [9 or 10] for the definitions) such that each fiber $A_\omega = p^{-1}(\{\omega\})$ is a C^* -algebra and multiplication and involution is continuous on E . The section algebra $\Gamma_0(E)$ is the set of all continuous functions $a: \Omega \rightarrow E; \omega \rightarrow a(\omega)$ such that $p(a(\omega)) = \omega$ for each $\omega \in \Omega$ and $\omega \rightarrow \|a(\omega)\|$ vanishes at infinity. The irreducible representations of $\Gamma_0(E)$ are given by the collection of the irreducible representations of the fibers A_ω , and the projection $P: \text{Prim}(A) \rightarrow \Omega$ which maps a primitive ideal to ω if and only if $J = \ker \rho$ for some $\rho \in \hat{A}_\omega$ is always continuous, open, and surjective. On the other side, if A is any C^* -algebra such that there exists a continuous and open map onto the locally compact space Ω , then there exists a C^* -bundle $p: E \rightarrow \Omega$ such that $A_\omega = A/\ker P^{-1}(\{\omega\})$ and $\Gamma_0(E)$ is isomorphic to A . These results were proved by Lee in [15]. Hence in this section we always assume that A is the section algebra $\Gamma_0(E)$ for a C^* -bundle $p: E \rightarrow \Omega$.

If $p: E \rightarrow \Omega$ is a C^* -bundle and $q: \tilde{\Omega} \rightarrow \Omega$ is a continuous map then the bundle pullback q^*E of E over q is defined to be the set of all $(\tilde{\omega}, e) \in \tilde{\Omega} \times E$ such that $q(\tilde{\omega}) = p(e)$, together with the obvious projection from q^*E onto

$\tilde{\Omega}$. The sections of q^*E can be identified with the continuous functions $f: \tilde{\Omega} \rightarrow E$ such that $p(f(\tilde{\omega})) = q(\tilde{\omega})$ for all $\tilde{\omega} \in \tilde{\Omega}$. If A is a locally closed (hence locally compact) subset of Ω , and $i: A \rightarrow \Omega$ the inclusion, then i^*E can be identified with $p^{-1}(A)$ and we denote i^*E always by $E|A$. Note that the set $\{a|A; a \in \Gamma_0(E)\}$ forms a dense subalgebra of $\Gamma_0(E|A)$ if A is closed in Ω . It should also be noted that, if U is an open subset of Ω , there exists a canonical short exact sequence

$$0 \rightarrow \Gamma_0(E|U) \rightarrow \Gamma_0(E) \rightarrow \Gamma_0(E|\Omega \setminus U) \rightarrow 0.$$

We start our investigations with the definition of subgroup actions on section algebras.

DEFINITION 3. Let $p: E \rightarrow \Omega$ be a C^* -bundle, G a locally compact group, and $H: \Omega \rightarrow \mathcal{K}(G)$ a continuous map. An *action of Ω^H on $A = \Gamma_0(E)$* is a family $\alpha = (\alpha_\omega)_{\omega \in \Omega}$ such that

- (1) each α_ω is a strongly continuous action of H_ω on A_ω ,
- (2) the map $\Omega^H \rightarrow E; (\omega, x) \rightarrow \alpha_\omega(x)(a(\omega))$ is continuous for each $a \in \Gamma_0(E)$.

If A is a locally closed subset of Ω and α an action of Ω^H on $\Gamma_0(E)$, then $\alpha_A = (\alpha_\omega)_{\omega \in A}$ is an action of A^H on $\Gamma_0(E|A)$ and is called the *restricted action of α* .

It is useful to give also a definition of an action of Ω^H on a topological space M . For this suppose that $P: M \rightarrow \Omega$ is an open continuous and surjective map. If W is a subset of Ω^H and $U \subseteq M$ we denote by $W * U$ the set $\{((\omega, x), m) \in W \times U; P(m) = \omega\}$.

DEFINITION 4. Let Ω^H and M be as above. An *action of Ω^H on M* is a family $\beta = (\beta_\omega)_{\omega \in \Omega}$ such that β_ω is a homomorphism from H_ω into the homeomorphism group of $M_\omega = P^{-1}(\{\omega\})$ such that the map

$$\Xi: \Omega^H * M \rightarrow M; ((\omega, x), m) \rightarrow \beta_\omega(x)(m)$$

is continuous.

We now see that actions on spaces are in a certain sense dual to actions on C^* -algebras.

PROPOSITION 6. Let α be an action of Ω^H on $A = \Gamma_0(E)$. For each $(\omega, x) \in \Omega^H$ and $\rho \in \hat{A}_\omega$ let

$$\beta_\omega(x)(\rho) = \rho \circ \alpha_\omega(x^{-1}).$$

Then $\beta = (\beta_\omega)_{\omega \in \Omega}$ is an action of Ω^H on \hat{A} . Conversely, if M is locally compact and β an action of Ω^H on M , then

$$(\alpha_\omega(x)(f| M_\omega))(m) = f| M_\omega(\beta_\omega(x^{-1})(m))$$

defines an action $\alpha = (\alpha_\omega)_{\omega \in \Omega}$ of Ω^H on $C_0(M)$.

Proof. To simplify notation let us denote $\beta_\omega(x)(\pi)$ by $(\omega, x) \cdot \pi$. So let $(\omega_0, x_0) \in \Omega^H$ and $\pi \in \hat{A}_\omega$. A typical neighborhood for $(\omega_0, x_0) \cdot \pi$ is given by

$$V = \{\rho \in \hat{A}; \|\rho(a)\| > \|(\omega_0, x_0) \cdot \pi(a)\| - \varepsilon\},$$

where $\varepsilon > 0$ and $a \in A$ such that $(\omega_0, x_0) \cdot \pi(a) \neq 0$. Let $b \in A$ such that $b(\omega_0) = \alpha_{\omega_0}(x_0^{-1})(a(\omega_0))$, $U = \{\rho \in \hat{A}; \|\rho(b)\| > \|\pi(b)\| - \varepsilon/2\}$, and W is a neighborhood of (ω_0, x_0) in Ω^H such that $\|\alpha_\omega(x^{-1})(a(\omega)) - b(\omega)\| < \varepsilon/2$ for all $(\omega, x) \in W$. Then

$$\begin{aligned} \|(\omega, x) \cdot \rho(a)\| &\geq \|\rho(b)\| - \|\rho(\alpha_\omega(x^{-1})(a(\omega))) - \rho(b(\omega))\| \\ &> \|\pi(b)\| - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = \|(\omega_0, x_0) \cdot \pi(a)\| - \varepsilon, \end{aligned}$$

for all $(\omega, x) \in W$ and $\rho \in U \cap \hat{A}_\omega$. Hence $\Xi(W * U) \subseteq V$ which proves that β is an action of Ω^H on \hat{A} .

Now let us suppose that M is a locally compact space and β an action of Ω^H on M . Let α be defined as in the proposition and let $f \in C_0(M)$. Suppose further that $(\omega_i, x_i)_{i \in I}$ is a net in Ω^H converging to (ω_0, x_0) , and let $g \in C_0(M)$ such that $g| M_{\omega_0} = \alpha_{\omega_0}(x_0)(f| M_{\omega_0})$. We have to show that $\|\alpha_{\omega_i}(x_i)(f| M_{\omega_i}) - g| M_{\omega_i}\| \rightarrow 0$ (see [10, Proposition 10.3]). Since $C_c(M)$ is dense in $C_0(M)$ we may assume that $f, g \in C_c(M)$. For each $i \in I$ we can find $m_i \in M_{\omega_i}$ such that

$$\|\alpha_{\omega_i}(x_i)(f| M_{\omega_i}) - g| M_{\omega_i}\| = |f((\omega_i, x_i^{-1}) \cdot m_i) - g(m_i)|.$$

If this expression is equal to zero for almost all $i \in I$ we are done. If not we can assume by passing to a subnet if necessary that this expression is greater than 0 for all $i \in I$. Then $m_i \in \Xi((\omega_i, x_i)_{i \in I} * \text{supp } f) \cup \text{supp } g$ which is relatively compact by the continuity of Ξ . Hence we can assume that $m_i \rightarrow m_0$ for some $m_0 \in M_{\omega_0}$, which implies that $(\omega_i, x_i) \cdot m_i \rightarrow (\omega_0, x_0) \cdot m_0$. Hence, for $\varepsilon > 0$, there exists an $i_0 \in I$ such that

$$|f((\omega_i, x_i) \cdot m_i) - f((\omega_0, x_0) \cdot m_0)| < \frac{\varepsilon}{2}$$

and

$$|g(m_i) - g(m_0)| < \frac{\varepsilon}{2},$$

for all $i > i_0$. But this finishes the proof since $g(m_0) = f((\omega_0, x_0) \cdot m_0)$ by the choice of g . ■

The following is our motivating example for defining subgroup actions. We see later that every action is restricted from an action as given in the example below.

EXAMPLE 2. Let (Ω, R) be an open regularization of the covariant system (G, A) . Then $A = \Gamma_0(E)$ for a C^* -bundle $p: E \rightarrow \Omega$. Suppose further that the stabilizer map $S: \Omega \rightarrow \mathcal{K}(G)$ is continuous. The action of G on $\Gamma_0(E)$ induces an action α_ω of S_ω on A_ω for each $\omega \in \Omega$, and it follows from [10, Proposition 10.3] that $\alpha = (\alpha_\omega)_{\omega \in \Omega}$ is an action of Ω^S on $\Gamma_0(E)$.

We are now following the constructions given by Raeburn and Williams in [24, Section 2] to define the crossed product $\Omega^H \times_x A$ by an action α of Ω^H on $A = \Gamma_0(E)$. Note that this construction was given in [24] for the special case of Example 2 where in addition A_ω was assumed to be primitive for all $\omega \in \Omega$. But a careful look shows that these assumptions are nowhere needed in the construction of the crossed product.

Let $q: \Omega^H \rightarrow \Omega$ be the canonical projection, and let $\Gamma_c(q^*E)$ be the set of all continuous sections of q^*E with compact support in Ω^H . Recall again that we can identify the sections of q^*E with the continuous functions $f: \Omega^H \rightarrow E$ such that $f(\omega, x) \in A_\omega$ for all $(\omega, x) \in \Omega^H$. We define multiplication, involution, and norm on $\Gamma_c(q^*E)$ by

$$\begin{aligned} f * g(\omega, x) &= \int_{H_\omega} f(\omega, z) \alpha_\omega(z)(g(\omega, z^{-1}x)) d_{H_\omega} z, \\ f^*(\omega, x) &= \Delta_G(x^{-1}) \alpha_\omega(x)(f(\omega, x^{-1})^*), \end{aligned}$$

and

$$\|f\|_I = \sup_{\omega \in \Omega} \int_{H_\omega} \|f(\omega, x)\| d_{H_\omega} x.$$

Following the arguments used in [24] we see that the completion of $\Gamma_c(q^*E)$ with respect to $\|\cdot\|_I$ is a Banach $*$ -algebra and we define $\Omega^H \times_x A$ as the enveloping C^* -algebra of this completion. The following proposition follows easily from [24, Lemma 2.3].

PROPOSITION 7. *The dual space $(\Omega^H \times_x A)^\wedge$ is in a one to one correspondence to the set of all pairs (ω, π) , $\omega \in \Omega$ and $\pi \in C^*(H_\omega, A_\omega)^\wedge$. This correspondence is given in one direction by $(\omega, \pi)(f) = \pi(f(\omega, \cdot))$ for $f \in \Gamma_c(q^*E)$.*

Before we proceed we want to define twisting maps for actions of Ω^H on section algebras. For this let N_τ be a closed normal subgroup of G such that $N_\tau \subseteq H_\omega$ for all $\omega \in \Omega$. A twisting map τ for the action of Ω^H on $A = \Gamma_0(E)$ is a strictly continuous homomorphism from N_τ into the group of unitaries $\mathcal{U}(A)$ of $\mathcal{M}(A)$ with the following property: If $\tau_\omega: N_\tau \rightarrow \mathcal{U}(A_\omega)$ is defined by

$$\tau_\omega(n) a(\omega) = (\tau(n) a)(\omega), \quad a \in A, \quad n \in N_\tau,$$

then τ_ω is a twisting map for the action of H_ω on A_ω for all $\omega \in \Omega$. If in Example 2 τ is a twisting map for the action of G on A , then it is easily seen that τ is also a twisting map for the action of Ω^S on A . If A is a locally closed subset of Ω and τ is a twisting map for the action of Ω^H on $\Gamma_0(E)$, then $\tau_A: N_\tau \rightarrow \mathcal{U}(\Gamma_0(E|A))$ defined by

$$(\tau_A(n)b)(\omega) = \tau_\omega(n) b(\omega), \quad n \in N_\tau, \quad b \in \Gamma_0(E|A), \quad \omega \in A,$$

is a twisting map for the restricted action α_A of A^H on $\Gamma_0(E|A)$.

We now define the *twisted crossed product* $\Omega^H \times_{x, \tau} A$ of Ω^H and A with respect to τ as the quotient $(\Omega^H \times_x A)/I_\tau$, where I_τ denotes the intersection of all kernels of irreducible representations (ω, ρ) such that ρ is a representation of $C^*(H_\omega, A_\omega, \tau_\omega)$.

It is useful to give a somewhat different realization of the twisted crossed product $\Omega^H \times_{x, \tau} A$ following along the lines of [14, Section 1]. For this let $(d_{\dot{H}})_{\dot{H} \in \mathcal{H}(G/N_\tau)}$ be the smooth choice of Haar measures on $\mathcal{H}(G/N_\tau)$ given by the equations

$$\int_H f(x) d_H x = \int_{\dot{H}} \int_{N_\tau} f(x_n) dn d_{\dot{H}} \dot{x}$$

for every $f \in C_c(G)$. For simplicity we always denote the Haar measure on N_τ by dn . We define $\Gamma_c(q^*E, \tau)$ as the set of all continuous sections $f: \Omega^H \rightarrow E$ with compact support modulo N_τ such that

$$f(\omega, xn) = f(\omega, x) \tau_\omega(xn^{-1}x^{-1}).$$

We equip $\Gamma_c(q^*E, \tau)$ with multiplication, involution, and norm by

$$f \star g(\omega, x) = \int_{\dot{H}_\omega} f(\omega, z) \alpha_\omega(z)(g(\omega, z^{-1}x)) d_{\dot{H}_\omega} z,$$

$$f^*(\omega, x) = A_{\dot{H}_\omega}(\dot{x}^{-1}) \alpha_\omega(x)(f(x^{-1})^*),$$

and

$$\|f\|_{I_\tau} = \int_{\dot{H}_\omega} \|f(\omega, x)\| d\dot{H}_\omega.$$

If we define $\Phi: \Gamma_c(q^*E) \rightarrow \Gamma_c(q^*E, \tau)$ by

$$(\Phi f)(\omega, x) = \int_{N_\tau} f(\omega, xn) \tau_\omega(xn x^{-1}) dn,$$

then it is easily seen that the irreducible $*$ -representations of the completion, say B , of $\Gamma_c(q^*E, \tau)$ are given by $\pi \circ \Phi$ where π runs through all irreducible representations of $\Omega^H \times_x A$ which preserve τ . Hence Φ can be extended to a $*$ -homomorphism from $\Omega^H \times_x A$ onto the enveloping C^* -algebra $C^*(B)$ of B with kernel I_τ . Thus $C^*(B)$ is isomorphic to $\Omega^H \times_{x, \tau} A$.

The next proposition follows easily from Proposition 7 by standard arguments (see also [24, Lemma 6.2]).

PROPOSITION 8. *Let α be an action of Ω^H on $A = \Gamma_0(E)$, τ a twisting map for α , U an open subset of Ω , and $A = \Omega \setminus U$. Then there exists a short exact sequence*

$$0 \rightarrow U^H \times_{x_U, \tau_U} \Gamma_0(E|U) \rightarrow \Omega^H \times_{x, \tau} \Gamma_0(E) \rightarrow A^H \times_{x_A, \tau_A} \Gamma_0(E|A) \rightarrow 0.$$

It follows from Proposition 7 and the definition of the twisted crossed product given above that in the case of an open regularization (Ω, R) of a system (G, A, τ) the space \mathcal{S}_R can be identified with the dual space of $\Omega^S \times_{x, \tau} A$. On the other side, we have seen in the preceding section that \mathcal{S}_R is also the dual space of the quotient $C^*(\Omega^S, A, \tau)/I_{\mathcal{S}_R}$. In fact, we have

PROPOSITION 9. *Suppose that (Ω, R) is an open regularization of the twisted covariant system (G, A, τ) such that the stabilizer map $S: \Omega \rightarrow \mathcal{K}(G)$ is continuous. Let $p: E \rightarrow \Omega$ be the C^* -bundle defined by R such that $A = \Gamma_0(E)$ and let α be the action of Ω^S on $\Gamma_0(E)$ as given in Example 2. Then $\Omega^S \times_{x, \tau} A$ is isomorphic to $C^*(\Omega^S, A, \tau)/I_{\mathcal{S}_R}$, where $I_{\mathcal{S}_R} = \bigcap \{\ker(\omega, \rho); (\omega, \rho) \in \mathcal{S}_R\}$. Furthermore, if A is a locally closed subset of Ω then $A^S \times_{x_A, \tau_A} \Gamma_0(E|A)$ is isomorphic to $C^*(A^S, A, \tau)/I_A$, where $I_A = \bigcap \{\ker(\omega, \rho); (\omega, \rho) \in \mathcal{S}_R, \omega \in A\} \subseteq C^*(A^S, A, \tau)$.*

Proof. We may assume that τ is trivial. We define $\Phi: C_c(\Omega^S, A) \rightarrow \Gamma_c(q^*E)$ by $\Phi f(x, \omega) = f(x, \omega)(\omega)$. Straightforward computations show that Φ preserves multiplication and involution. If $g \in C_c(\Omega^S)$ and $a \in \Gamma_0(E)$,

then $\Phi(g \otimes a)(\omega, x) = g(\omega, x) a(\omega)$ and it is easily seen that these functions form a dense subset in $\Gamma_c(q^*E)$. Now let $(\omega, \rho) \in \mathcal{S}_R$. Then

$$(\omega, \rho)(f) = \int_{S_\omega} \rho_A(f(\omega, x)) \rho_{S_\omega}(x) d_{S_\omega} x,$$

and since $\ker R^{-1}(\{\omega\}) \subseteq \ker \rho_A$ this is equal to

$$\int_{S_\omega} \rho_A(f(\omega, x)(\omega)) \rho_{S_\omega}(x) d_{S_\omega} x = (\omega, \rho)(\Phi f).$$

But this shows that Φ can be extended to all of $C^*(\Omega^S, A)$ and that $\ker \Phi = I_{\mathcal{S}_R}$. The last statement of the proposition follows from the fact that the map $\mathcal{S}_R \rightarrow \Omega$; $(\omega, \rho) \rightarrow \omega$ is continuous. ■

Remark 2. Let (G, A, τ) be a twisted covariant system, Ω a locally compact space, and $H: \Omega \rightarrow \mathcal{K}(G)$ a continuous map such that $N_\tau \subseteq H_\omega$ for all $\omega \in \Omega$. We can define an action α of Ω^H on $C_0(\Omega, A)$ and a twisting map for this action in the following way: For each $\omega \in \Omega$ let α_ω be the restriction of the action of G on A to H_ω , and define $\tilde{\tau}: N_\tau \rightarrow \mathcal{U}(C_0(\Omega, A))$ by

$$(\tilde{\tau}(n)\varphi)(\omega) = \tau(n)(\varphi(\omega)),$$

$n \in N_\tau$, $\omega \in \Omega$, and $\varphi \in C_0(\Omega, A)$. It follows almost from the definitions and the description of the dual spaces that $\Omega^H \times_{\alpha, \tilde{\tau}} C_0(\Omega, A)$ is isomorphic to $C^*(\Omega^H, A, \tau)$. Hence our subgroup algebras defined before are special cases of the twisted crossed products by subgroup actions on section algebras.

If H is a closed subgroup of G and (H, A) a covariant system, then there is a well-known construction of the induced C^* -algebra $\text{Ind}_H^G A$ with a canonical action of G . In the case that A is commutative, i.e., in the case of transformation groups, this procedure is rather classical. For non-commutative A , $\text{Ind}_H^G A$ was defined in [23] in the more general setting where G is replaced by an arbitrary locally compact space such that H acts properly and freely on this space. We now define induced systems in which A is a section algebra with base space Ω and H is replaced by a continuous map $H: \Omega \rightarrow \mathcal{K}(G)$ and an action of Ω^H on A .

DEFINITION 5. Let $A = \Gamma_0(E)$ be the section algebra of the C^* -bundle $p: E \rightarrow \Omega$ and $H: \Omega \rightarrow \mathcal{K}(G)$ a continuous map. Suppose further that $\alpha = (\alpha_\omega)_{\omega \in \Omega}$ is an action of Ω^H on A . We define

$$\text{Ind}_{\Omega^H}^G A = \{F \in C^b(G, A); \alpha_\omega(h)(F(x)(\omega)) = F(xh^{-1})(\omega)$$

$$\text{for all } \omega \in \Omega \text{ and } h \in H_\omega \text{ such that } (\omega, \dot{x}) \rightarrow \|F(x)(\omega)\|$$

$$\text{defines an element of } C_0(\Omega \times_H G)\}.$$

$\text{Ind}_{\Omega^H}^G A$ equipped with the pointwise operations and the supremum-norm is called the *induced C^* -algebra* of $\Gamma_0(E)$ by the action of Ω^H . There is a canonical action of G on $\text{Ind}_{\Omega^H}^G A$ given by ${}^x F(y) = F(x^{-1}y)$, which is called the induced action of G on $\text{Ind}_{\Omega^H}^G A$.

Since $\text{Ind}_{\Omega^H}^G A$ is invariant under pointwise multiplication with functions in $C_0(\Omega \times_H G)$, it follows that the elements $F \in \text{Ind}_{\Omega^H}^G A$ which have compact support in $\Omega \times_H G$ are dense in $\text{Ind}_{\Omega^H}^G A$. Using the usual compactness arguments, this implies that the induced action of G on $\text{Ind}_{\Omega^H}^G A$ is strongly continuous. Furthermore, if τ is a twisting map for the action of Ω^H on A , then we can define the induced twisting map $\text{Ind } \tau: N_\tau \rightarrow \mathcal{U}(\text{Ind}_{\Omega^H}^G A)$ by

$$(\text{Ind } \tau(n) F)(x) = \tau(x^{-1}nx)(F(x)), \quad n \in N_\tau, \quad x \in G.$$

Straightforward calculations show that $\text{Ind } \tau$ is a twisting map for $(G, \text{Ind}_{\Omega^H}^G A)$ (this was done for the case of ordinary induced systems in [30]). The resulting twisted covariant system $(G, \text{Ind}_{\Omega^H}^G A, \text{Ind } \tau)$ is called the *induced twisted covariant system*.

We next investigate some properties of induced systems. Since the proofs of some of the results are nearly the same as in the case of ordinary induced systems, we sometimes give only sketches of the proofs or refer to the proofs in the ordinary case. The first result describes the dual spaces of the induced C^* -algebra $\text{Ind}_{\Omega^H}^G A$ and gives some relations between certain properties of A and $\text{Ind}_{\Omega^H}^G A$. Recall that the open projection $P: \text{Prim}(A) \rightarrow \Omega$ is given by $P(\ker \rho) = \omega$ if and only if $\rho \in \hat{A}_\omega$. Let us again denote the dual action of $(\omega, h) \in \Omega^H$ on $\rho \in \hat{A}_\omega$ by $(\omega, h) \cdot \rho$. We define equivalence relations \sim and $\dot{\sim}$ on $G \times \hat{A}$ and $G \times \text{Prim } A$, respectively, by

$$(x, \pi) \sim (y, \rho) \Leftrightarrow P(\ker \pi) = P(\ker \rho) = \omega \text{ and there exists } h \in H_\omega$$

$$\text{such that } (x, \pi) = (yh^{-1}, (\omega, h) \cdot \rho),$$

and

$$(x, \ker \pi) \dot{\sim} (y, \ker \rho) \Leftrightarrow P(\ker \pi) = P(\ker \rho) = \omega \text{ and there exists } h \in H_\omega$$

$$\text{such that } (x, \ker \pi) = (yh^{-1}, \ker((\omega, h) \cdot \rho)).$$

PROPOSITION 10. *For each $x \in G$ and $\pi \in \hat{A}$ let $M(x, \pi)$ denote the representation of $\text{Ind}_{\Omega^H}^G A$ defined by $M(x, \pi)(F) = \pi(F(x))$. Then $M: G \times \hat{A} \rightarrow (\text{Ind}_{\Omega^H}^G A)^\wedge$; $(x, \pi) \rightarrow M(x, \pi)$ is a continuous open and surjective map, which implements a homeomorphism between $(G \times \hat{A})/\sim$ and $(\text{Ind}_{\Omega^H}^G A)^\wedge$. Similarly, $\text{Prim}(\text{Ind}_{\Omega^H}^G A)$ is homeomorphic to $(G \times \text{Prim}(A))/\dot{\sim}$. Furthermore,*

- (1) $\text{Prim}(A)$ is Hausdorff if and only if $\text{Prim}(\text{Ind}_{\Omega^H}^G A)$ is Hausdorff.
- (2) A is of type I if and only if $\text{Ind}_{\Omega^H}^G A$ is of type I.
- (3) A has continuous trace if and only if $\text{Ind}_{\Omega^H}^G A$ has continuous trace.

Proof. The first assertion and the surjectivity of M follow by arguments similar to those used in the proof of [23, Lemma 2.6] if we can show that the set $\{F(x)(\omega); F \in \text{Ind}_{\Omega^H}^G A\}$ is dense in A_ω for all $\omega \in \Omega$, $x \in G$, and if we replace Ω/G in the setting of [23] by $\Omega \times_H G$ in our setting. By the construction of $\text{Ind}_{\Omega^H}^G A$ it is enough to show the density for $x = e$. For this let $\omega_0 \in \Omega$, $a \in A$, and V be a small neighborhood of e in H_{ω_0} . Then we can find $g \in C_c(\Omega \times G)$ such that $g(\omega_0, x) \neq 0$ only if $x \in V$ and $\int_{H_{\omega_0}} g(\omega_0, x) d_{H_{\omega_0}} x = 1$. Then standard arguments show that $F(e)(\omega_0)$ is closed to $a(\omega_0)$ if $F \in \text{Ind } A$ is defined by

$$F(x)(\omega) = \int_{H_\omega} g(\omega, xh) \alpha_\omega(h)(a(\omega)) d_{H_\omega} h.$$

It is quite trivial that M is continuous, and the openness of M follows from nearly the same arguments as used in the proof of [23, Proposition 3.2], again by changing the role of the space Ω/G in the setting of [23] with $\Omega \times_H G$ in our setting. This shows that $(\text{Ind}_{\Omega^H}^G A)^\wedge$ is homeomorphic to $(G \times \hat{A})/\sim$, and it follows easily from this that $\text{Prim}(\text{Ind}_{\Omega^H}^G A)$ has a similar description.

To prove Assertions 1 to 3 note first that since A is a quotient of $\text{Ind}_{\Omega^H}^G A$ all three properties follow for A if they are true for $\text{Ind}_{\Omega^H}^G A$, and that the inverse for Assertion 2 follows immediately from the description of the irreducible representations of the induced algebra.

Let us now assume that $\text{Prim}(A)$ is Hausdorff. Then $\text{Prim}(A)$ is a locally compact Hausdorff space, and it follows from Proposition 6 that the action of Ω^H on $\text{Prim}(A)$ given by $(\omega, h) \cdot \ker \rho = \ker((\omega, h) \cdot \rho)$, $\rho \in \hat{A}_\omega$, induces an action $\tilde{\alpha}$ of Ω^H on $C_0(\text{Prim } A)$. The description of $(\text{Ind}_{\Omega^H}^G C_0(\text{Prim}(A)))^\wedge$ given above shows that this space is homeomorphic to $\text{Prim}(\text{Ind}_{\Omega^H}^G A)$, and since the induced algebra of a commutative C^* -algebra is also commutative it follows that $\text{Prim}(\text{Ind}_{\Omega^H}^G A)$ is Hausdorff.

Last but not least let us assume that A has continuous trace. Since we already know that $(\text{Ind}_{\Omega^H}^G A)^\wedge$ is Hausdorff, it is enough to show that for all $M(x, \pi) \in (\text{Ind}_{\Omega^H}^G A)^\wedge$ there exists an element $F \in \text{Ind}_{\Omega^H}^G A$ such that $M(y, \rho)(F)$ is a projection of rank one for all $M(y, \rho)$ in some neighborhood of $M(x, \pi)$. But this can easily be done with the same arguments as given in the proof of [23, Corollary 3.9] by using functional calculus. ■

The following theorem shows that induced systems are exactly those which have open σ -trivial regularizations.

THEOREM 3. *Suppose that τ is a twisting map for the action α of Ω^H on $A = \Gamma_0(E)$. Let $R: \text{Prim}(\text{Ind}_{\Omega^H}^G A) \rightarrow \Omega \times_H G$ be defined by $R(\ker M(x, \rho)) =$*

(ω, \dot{x}) if and only if $P(\ker \rho) = \omega$. Then $(\Omega \times_H G, R)$ is an open-trivial regularization for $(G, \text{Ind}_{\Omega^H}^G A, \text{Ind } \tau)$.

Conversely, if (Ω, R) is an open σ -trivial regularization of a twisted coariant system (G, A, τ) with continuous section Σ , then (G, A, τ) is isomorphic to the induced system $(G, \text{Ind}_{\Sigma^S}^G D, \text{Ind } \tau_\Sigma)$, where $D = A/\ker R^{-1}(\Sigma)$, $\alpha = (\alpha_\omega)_{\omega \in \Sigma}$ is the action of Σ^S on D induced by the action of G on A , and τ_Σ is the twisting map for α induced by τ . More precisely, the map

$$\Phi: A \rightarrow \text{Ind}_{\Sigma^S}^G D; \Phi(a)(x) = x^{-1}a + \ker R^{-1}(\Sigma)$$

is a G -equivariant isomorphism which intertwines τ and $\text{Ind } \tau_\Sigma$.

Proof. Let $R: \text{Prim}(\text{Ind}_{\Omega^H}^G A) \rightarrow \Omega \times_H G$ be defined as in the first part of the theorem. It is clear that R is G -equivariant continuous and surjective. To see that R is open, let $(\omega_i, \dot{x}_i) \rightarrow (\omega, \dot{x})$ in $\Omega \times_H G$ and let $\rho \in \hat{A}_\omega$. By the openness of the quotient map $\Omega \times G \rightarrow \Omega \times_H G$ and the map $P: \text{Prim}(A) \rightarrow \Omega$ we find, by passing to a subnet if necessary, a net $(h_i)_{i \in I}$ in G with $h_i \in H_{\omega_i}$ for each $i \in I$, and a net $(\rho_i)_{i \in I} \subseteq \hat{A}$ with $\rho_i \in \hat{A}_{\omega_i}$ such that $(\omega_i, x_i h_i) \rightarrow (\omega, x)$ in $\Omega \times G$ and $\rho_i \rightarrow \rho$ in \hat{A} . Hence $M(x_i h_i, \rho_i) \rightarrow M(x, \rho)$ in $(\text{Ind}_{\Omega^H}^G A)^\wedge$ and $R(\ker M(x_i h_i, \rho_i)) = (\omega_i, \dot{x}_i)$ for all $i \in I$. This proves the first part of the theorem since $\Omega \times_H G$ is a σ -trivial G -space by Proposition 5.

The proof of the second part was given in [3] in the case of ordinary induced covariant systems and a trivial twisting map. It is quite easy to modify the proof to show that Φ is a G -equivariant isomorphism. Hence it remains to show that Φ intertwines τ and $\text{Ind } \tau_\Sigma$. For this let $a \in A$ and $n \in N_\tau$. Then, with $I_\Sigma = R^{-1}(\Sigma)$ we get

$$\begin{aligned} (\text{Ind } \tau_\Sigma(n) \Phi(a))(x) &= \tau_\Sigma(x^{-1}nx)(x^{-1}a + I_\Sigma) = (\tau(x^{-1}nx) x^{-1}a) + I_\Sigma \\ &= x^{-1}(\tau(n)a) + I_\Sigma = (\Phi(\tau(n)a))(x), \end{aligned}$$

for all $x \in G$. ■

COROLLARY 3. Let $H: \Omega \rightarrow \mathcal{K}(G)$ be a continuous map, α an action of Ω^H on the C^* -algebra $A = \Gamma_0(E)$ and τ a twisting map for α . Then $\Omega^H \times_{\alpha, \tau} A$ is Morita-equivalent to $C^*(G, \text{Ind}_{\Omega^H}^G A, \text{Ind } \tau)$.

Proof. The proof follows immediately from Theorem 2, Proposition 9, and the first part of the theorem above. ■

We now show that, in the case that H_ω/N_τ is amenable for all $\omega \in \Omega$, $\Omega^H \times_{\alpha, \tau} A$ itself is a section algebra of a C^* -bundle $r: \tilde{E} \rightarrow \Omega$. This is a corollary of the following proposition. Recall that a $*$ -representation π is called weakly contained in a $*$ -representation ρ , denoted $\pi < \rho$, iff $\ker \pi \supseteq \ker \rho$.

PROPOSITION 11. *Let (Ω, R) be a regularization of (G, A, τ) such that the stabilizer map $S: \Omega \rightarrow \mathcal{K}(G)$ is continuous, R is open and surjective, and S_ω/N_τ is amenable for all $\omega \in \Omega$. Then the projection $P: \mathcal{S}_R \rightarrow \Omega; (\omega, \rho) \rightarrow \omega$ is continuous and open.*

Proof. It was shown in [4] that P is continuous. In order to prove that P is open let $(\omega_i)_{i \in I}$ be a net in Ω converging to some ω_0 , and let $(\omega_0, \rho_0) \in \mathcal{S}_R$. We have to show the existence of a subnet $((\omega_l))_{l \in L}$ and a related net $((\omega_l, \rho_l))_{l \in L} \subseteq \mathcal{S}_R$, such that $(\omega_l, \rho_l) \rightarrow (\omega_0, \rho_0)$. For each $i \in I$ let $\pi_i \in \text{Rep}(A)$ such that $\ker \pi_i = \ker R^{-1}(\{\omega_i\})$ and let $\pi_0 \in \text{Rep}(A)$ such that $\ker \pi_0 = \ker R^{-1}(\{\omega_0\})$. We claim that $\pi_i \rightarrow \pi_0$ in $\text{Rep}(A)$. By [8, Proposition 1.2] $\pi_i \rightarrow \pi_0$ if and only if π_0 is weakly contained in $(\pi_i)_{i \in I}$ for every subnet $(\pi_l)_{l \in L}$ of $(\pi_i)_{i \in I}$. So let $(\pi_l)_{l \in L}$ be such a subnet. Since R is open, we can find, by passing to another subnet if necessary, for all $\sigma \in \hat{A}$ with $\sigma < \pi_0$ a net $(\sigma_l)_{l \in L} \subseteq \hat{A}$ such that $\sigma_l \rightarrow \sigma$ in \hat{A} and $\sigma_l < \pi_l$ for all $l \in L$. But this implies

$$\bigcap_{l \in L} \ker \pi_l \subseteq \bigcap_{l \in L} \ker \sigma_l \subseteq \ker \sigma.$$

Since this is true for all $\sigma \in \hat{A}$ which are weakly contained in π_0 it follows that $\pi_0 < (\pi_l)_{l \in L}$. This proves the claim.

Since $\rho_{A_0} < \pi_0$ it follows now from [8, Proposition 1.3] that $\pi_i \rightarrow \rho_{A_0}$ in $\text{Rep}(A)$. By the continuity of inducing representations [4, Corollary 2] we get $(\omega_i, \text{ind}_{N_\tau}^{S_{\omega_i}} \pi_i) \rightarrow (\omega_0, \text{ind}_{N_\tau}^{S_{\omega_0}} \rho_{A_0})$ in $\text{Rep}(C^*(\Omega^S, A, \tau))$, where we identify the representations of A with those of $C^*(N_\tau, A, \tau)$ (recall that the latter algebra is canonically isomorphic to A). Since S_{ω_0}/N_τ is amenable, it follows that $\rho_0 < \text{ind}_{N_\tau}^G \pi_0$. Hence, by passing to a subnet if necessary, we find a net $((\omega_i, \rho_i))_{i \in I} \subseteq C^*(\Omega^S, A, \tau)^\wedge$ which converges to (ω_0, ρ_0) such that $\rho_i < \text{ind}_{N_\tau}^{S_{\omega_i}} \pi_i$ for all $i \in I$. But this implies that $\rho_{A_i} < \pi_i$ for all $i \in I$ since $\ker \pi_i$ is invariant under the action of S_{ω_i} and therefore $\ker \pi_i = \ker((\text{ind}_{N_\tau}^{S_{\omega_i}} \pi_i)|N_\tau)$. This shows that $(\omega_i, \rho_i) \in \mathcal{S}_R$ for all $i \in I$ and the proposition is proved. ▀

COROLLARY 4. *Let τ be a twisting map for an action α of Ω^H on $A = \Gamma_0(E)$, and suppose that H_ω/N_τ is amenable for all $\omega \in \Omega$. Then $Q: \text{Prim}(\Omega^H \times_{\alpha, \tau} A) \rightarrow \Omega; \ker(\omega, \rho) \rightarrow \omega$ is continuous open and surjective. In particular, $\Omega^H \times_{\alpha, \tau} A$ is isomorphic to the section algebra of a C^* -bundle $q: F \rightarrow \Omega$ with fibers isomorphic to $C^*(H_\omega, A_\omega, \tau_\omega)$.*

Proof. Let $(G, \text{Ind}_{\Omega^H}^G A, \text{Ind } \tau)$ be the induced system. Then $(\Omega \times_H G, R)$, where $R: \text{Prim}(\text{Ind}_{\Omega^H}^G A) \rightarrow \Omega \times_H G$ is defined as in Theorem 3, is an open and surjective regularization of this system. Hence it follows from Proposition 11 that the projection P from \mathcal{S}_R onto $\Omega \times_H G$ is open. Let us identify Ω with the closed set $\{(\omega, \dot{e}); \omega \in \Omega\} \subseteq \Omega \times_H G$.

By Proposition 9 we know that $(\Omega^H \times_{\alpha, \tau} A)^\wedge$ is homeomorphic to $P^{-1}(\Omega)$. Thus the restriction of P to $\Omega^H \times_{\alpha, \tau} A$ is open, too. Hence $\tilde{P}: \text{Prim}(\Omega^H \times_{\alpha, \tau} A) \rightarrow \Omega$ given by $\tilde{P}(\ker(\omega, \rho)) = \omega$ is also open and continuous. The rest of the Corollary follows from Lee's Theorem [15, Theorem 4]. ■

This result has an interesting corollary itself. We consider the very special case of Fell's subgroup algebra $C^*(\mathcal{K}(G)^{\text{Id}})$, taking $A = \mathbb{C}$ and the identity Id on $\Omega = \mathcal{K}(G)$. As we have seen in Remark 2, this is a special case of a crossed product by a subgroup action. Fell has defined two topologies on $\mathcal{K}(G)$, namely the compact-open topology we have always used here and the *representational* topology (for the definition see [8, p. 431]). It was shown by Fell that these topologies coincide whenever G is either compact, discrete, or abelian. The proof of the following corollary follows from Corollary 4 and [8, Theorem 2.1].

COROLLARY 5. *Let G be an amenable locally compact group. Then the representational and the compact-open topologies on $\mathcal{K}(G)$ coincide.*

We finish this section with an example which shows that regularizations might be useful in the study of the structure of crossed products even in the case of commutative A .

EXAMPLE 3. We define an action of \mathbb{R} on \mathbb{C}^2 by $t(z_1, z_2) = (e^{2\pi i t/|z_1|} z_1, e^{2\pi i t} z_2)$. It is clear that the identity map on \mathbb{C}^2 is a regularization for the system $(\mathbb{R}, C_0(\mathbb{C}^2))$ since \mathbb{R} is amenable and \mathbb{R} and \mathbb{C}^2 are second countable, but it is also clear that this map is not a complete regularization and the stabilizer map is not continuous. In fact, it is easily seen that the stabilizer map is non-continuous in every $(z_1, z_2) \in \mathbb{C}^2$ such that $|z_2|$ is a rational number. It follows that the usual Mackey analysis can describe the primitive ideal space of $C^*(\mathbb{R}, C_0(\mathbb{C}^2))$ but gives no further information about the structure of this algebra.

We now define a map $R: \mathbb{C}^2 \rightarrow \mathbb{C} \times \mathbb{R}^+$ by $R((z_1, z_2)) = (z_1, |z_2|)$. Then $(\mathbb{C} \times \mathbb{R}^+, R)$ is easily seen to be an open σ -trivial regularization of $(\mathbb{R}, C_0(\mathbb{C}^2))$ if we define the action of \mathbb{R} on $\mathbb{C} \times \mathbb{R}^+$ by $t(z_1, s) = (e^{2\pi i t/|z_1|} z_1, s)$ (compare with [24, Example 4.6]). A continuous section for this action on $\mathbb{C} \times \mathbb{R}^+$ is given by $\Sigma = (\mathbb{R}^+)^2$. Hence it follows from Theorem 3 that $C^*(\mathbb{R}, C_0(\mathbb{C}^2))$ is Morita-equivalent to $\Sigma^S \times_\alpha A$, where $A = C_0(\mathbb{C} \times \mathbb{R}^+)$ is a section algebra with base space Σ and fibers $A_{(r,s)} = C(\mathbb{T})$ if $r \neq 0$ and $A_{(0,s)} = \mathbb{C}$, $r, s \in \mathbb{R}^+$. The stability groups $S_{(r,s)}$ are equal to $r\mathbb{Z}$ if $r \neq 0$ and \mathbb{R} if $r = 0$ and they act on $A_{(r,s)}$ by multiplication of $z \in \mathbb{T}$ with $e^{2\pi i t}$ if $r \neq 0$, $t \in S_{(r,s)}$, and obviously trivially if $r = 0$. Thus it follows from Corollary 4 that $\Sigma^S \times_\alpha A$ is isomorphic to a section algebra $\Gamma_0(E)$ of a C^* -bundle $p: E \rightarrow (\mathbb{R}^+)^2$ with fibers

- (1) $B_{(r,s)} = C^*(r\mathbb{Z}, C(\mathbb{T}))$, if $r, s > 0$,
- (2) $B_{(r,0)} = C^*(r\mathbb{Z}) = C(\mathbb{T})$, if $r \neq 0$,
- (3) $B_{(0,s)} = C^*(\mathbb{R}, C(\mathbb{T}))$, if $s \neq 0$, and
- (4) $B_{(0,0)} = C^*(\mathbb{R}) = C_0(\mathbb{R})$.

Note that the $B(r, s)$ for $r, s > 0$ are rational or irrational rotation algebras depending on whether r is rational or irrational. These algebras have been studied extensively in the literature (see [27] for the irrational case). The primitive ideal spaces of the fibers can be computed as follows:

- (1) $\text{Prim}(B_{(r,s)}) = \{\text{pt}\}$, if $r, s > 0$ and r is irrational,
- (2) $\text{Prim}(B_{(r,s)}) = \hat{B}_{(r,s)} = \mathbb{T}^2$, if $r, s > 0$ and r is rational,
- (3) $\text{Prim}(B_{(r,0)}) = \hat{B}_{(r,0)} = \mathbb{T}$, if $r > 0$,
- (4) $\text{Prim}(B_{(0,s)}) = \mathbb{T}$, if $s > 0$, and
- (5) $\text{Prim}(B_{(0,0)}) = \hat{B}_{(0,0)} = \mathbb{R}$.

To compute the topology of $\text{Prim}(\Sigma^S \times_{\alpha} A)$ let us first denote by A_r the subset $\{r\} \times \mathbb{R}^+$ of $\Sigma = (\mathbb{R}^+)^2$. Then $A_r^S \times_{\alpha_r} \Gamma_0(E|A_r)$ is isomorphic to $C^*(r\mathbb{Z}, C_0(\mathbb{C}))$ if $r > 0$ and $C^*(\mathbb{R}, C_0(\mathbb{C}))$ if $r = 0$, where the action is given by multiplication with $e^{2\pi i t}$ for $t \in r\mathbb{Z}$ or $t \in \mathbb{R}$, respectively. It is very easy to compute the topology on the primitive ideal spaces of these fibers. This gives us the description of convergence of sequences $((r_n, s_n), J_n)_{n \in \mathbb{N}} \subseteq \text{Prim}(\Sigma^S \times_{\alpha} A)$ if $(r_n)_{n \in \mathbb{N}}$ is constant. If $A = \mathbb{R}^+ \times \{0\}$, then $\text{Prim}(A^S \times_{\alpha} \Gamma_0(E|A))$ is homeomorphic to $(\mathbb{R}^+ \times \mathbb{R})/\sim$, where \sim is the equivalence relation given by

$$(r, t) \sim (r', t') \Leftrightarrow \begin{cases} r = r' & \text{and} & t' \in t \frac{1}{r} \mathbb{Z} & \text{if } r \neq 0 \\ r = r' & \text{and} & t = t' & \text{if } r = 0. \end{cases}$$

Hence we know convergence of sequences if $s_n = 0$ for almost all $n \in \mathbb{N}$.

Now let $((r_n, s_n), J_n)_{n \in \mathbb{N}}$ be a sequence in $\text{Prim}(\Sigma^S \times_{\alpha} A)$ such that $(r_n)_{n \in \mathbb{N}}$ is not (almost) constant and $s_n \neq 0$ for all $n \in \mathbb{N}$. In order to have convergence of the sequence, it is necessary that $(r_n, s_n) \rightarrow (r, s)$ for some $(r, s) \in (\mathbb{R}^+)^2$. If $r \neq 0$, then our sequence converges to every pair $((r, s), J)$ with $J \in \text{Prim}(B_{(r,s)})$, and this does not depend on the choice of the J_n . This is an easy observation in the case where r_n is irrational for infinitely many n , but a bit more tricky if all r_n are rationals. In this case let $r_n = p_n/q_n$, the cancelled fraction. Since $(r_n)_{n \in \mathbb{N}}$ is assumed to be a non-constant convergent sequence it follows, by passing to a subsequence if necessary, that $q_n \rightarrow \infty$. Since $r \neq 0$ this implies also that $p_n \rightarrow \infty$. Hence we see that for every $z \in \mathbb{T}$ and any sequence $(z_n)_{n \in \mathbb{N}}$, z can be approximated by the

$r_n\mathbb{Z}$ orbits of the z_n in \mathbb{T} . Furthermore, the stability groups $p_n\mathbb{Z}$ of the action of $r_n\mathbb{Z}$ converge to $\{0\}$ in \mathbb{R} . These observations and the continuity of inducing representations of subgroup algebras [4, Proposition 6] give the prove of our statement for rational sequences.

Finally let us assume that $(r_n, s_n)_{n \in \mathbb{N}}$ is as above such that $r_n \rightarrow 0$ and $s_n \rightarrow s$. Again, if r_n is irrational for infinitely many $n \in \mathbb{N}$, then $((r_n, s_n), J_n)_{n \in \mathbb{N}}$ converges to every pair $((0, s), J)$ for every $J \in \text{Prim}(B_{(0, s)})$. The same is true if $(r_n)_{n \in \mathbb{N}}$ contains a subsequence p_n/q_n of cancelled fractions such that $p_n \rightarrow \infty$. Thus it remains to look to the case where $r_n = p/q_n$ for a constant number p . In this case we have that the stability group for the action of $r_n\mathbb{Z}$ on \mathbb{T} is constantly equal to $p\mathbb{Z}$. The parametrization of $\text{Prim}(B_{(r_n, s_n)})$ is given by $\widehat{p\mathbb{Z}} \times (\mathbb{T}/r_n\mathbb{Z})$ and the parametrization of $\text{Prim}(B_{(0, s)})$ is given by $\widehat{\mathbb{Z}} = \mathbb{T}$ if $s \neq 0$ and by $\widehat{\mathbb{R}} = \mathbb{R}$ if $s = 0$. Hence if we write J_n as $\chi_n \times [z_n]$ and if $\chi \in \widehat{\mathbb{Z}}$ or $\widehat{\mathbb{R}}$, respectively, we get $((r_n, s_n), (\chi_n, [z_n])) \rightarrow ((0, s), \chi)$ if and only if $\chi_n \rightarrow \chi_0$ for some $\chi_0 \in \widehat{p\mathbb{Z}}$ such that $\chi_0 = \chi|_{\mathbb{Z}}$.

We have now a complete description of the topology of the primitive ideal space of $\Sigma^S \times_x A$ and hence of $\text{Prim}(C^*(\mathbb{R}, C_0(\mathbb{C}^2)))$ by Morita-equivalence. As a consequence we see that $\text{Prim}(C^*(\mathbb{R}, C_0(\mathbb{C}^2)))$ contains no open Hausdorff subset which makes this space quite awful as a topological space. Note that in this example it is also possible to apply Williams' results in [31] to get a description of $\text{Prim}(\mathbb{R}, C_0(\mathbb{C}^2))$ as a certain quotient space of $\mathbb{C}^2 \times \mathbb{R}$, but we think it is more difficult to describe convergence in this quotient space. One should also remark that Morita-equivalence between two C^* -algebras is a much stronger relation than the relation of having homeomorphic primitive ideal spaces, and offers much more information about the structure of the C^* -algebras. We will see this more clearly in the following section.

5. TWISTED CROSSED PRODUCTS WITH CONTINUOUS TRACE

In this section we use our previous results to investigate whether a twisted crossed product $C^*(G, A, \tau)$ has continuous trace. We start our investigations with a general result.

THEOREM 4. *Let (Ω, R) be an open locally σ -trivial regularization of the twisted covariant system (G, A, τ) , and let α be the action of Ω^S on A induced by the action of G on A . If $\Omega^S \times_{\alpha, \tau} A$ has continuous trace then so does $C^*(G, A, \tau)$. Furthermore, if S_ω/N_τ is amenable for all $\omega \in \Omega$, then $C^*(G, A, \tau)$ has continuous trace if and only if $\Omega^S \times_{\alpha, \tau} A$ has continuous trace.*

Proof. Since Ω is a locally σ -trivial G -space we can find for each $\pi \in C^*(G, A, \tau)^\wedge$ a G -invariant open subset U in Ω such that U is a σ -trivial G -space and $\pi \in C^*(G, J, \tau_J)^\wedge$, where $J = \ker R^{-1}(\Omega \setminus U)$ and τ_J is the "restriction" of τ to J (compare Corollary 2). Analogous properties are true for the irreducible representations of $\Omega^S \times_{x, \tau} A$. Hence, since Ω/G is Hausdorff and the continuous trace property is a local property for C^* -algebras with Hausdorff dual space, we may assume that Ω itself is a σ -trivial G -space.

So let Σ be a continuous section for Ω/G and assume that $\Omega \times_{x, \tau} A$ has continuous trace. Since Σ is closed in Ω we know that $\Sigma^S \times_{x_\Sigma, \tau_\Sigma} D$, with $D = A/\ker R^{-1}(\Sigma) = \Gamma_0(E|\Sigma)$, is a quotient of $\Omega^S \times_{x, \tau} A$ and therefore has continuous trace, too. But $C^*(G, A, \tau)$ is Morita-equivalent to $\Sigma^S \times_{x_\Sigma, \tau_\Sigma} D$. Hence $C^*(G, A, \tau)$ has continuous trace if and only if the same is true for $\Sigma^S \times_{x_\Sigma, \tau_\Sigma} D$ by [32, Theorem 2.15].

To complete the proof it remains to show that $\Sigma^S \times_{x_\Sigma, \tau_\Sigma} D$ having continuous trace implies that $\Omega^S \times_{x, \tau} A$ has continuous trace if S_ω/N_τ is amenable for all $\omega \in \Omega$. For this we define an action of G on $\Omega^S \times_x A$ given on $f \in \Gamma_c(q^*E)$ by the equation

$${}^x f(\omega, s) = \delta(\omega, x) {}^x (f(x^{-1}\omega, x^{-1}sx)),$$

$(\omega, s) \in \Omega^S$, $x \in G$, where $\delta: \Omega \times G \rightarrow \mathbb{R}^+$ is given by

$$\delta(\omega, x) = \left(\int_{S_{x^{-1}\omega}} g(s) d_{S_{x^{-1}\omega}} s \right) \cdot \left(\int_{S_\omega} g(x^{-1}sx) d_{S_\omega} s \right)^{-1}$$

for any $g \in C_0^+(G)$ such that $g(e) \neq 0$ (compare with [4, Section 4]). The resulting action of G on $(\Omega^S \times_x A)^\wedge$ is given by ${}^x(\omega, \rho) = (x\omega, {}^x\rho)$. Since the set of irreducible representations which preserve τ is invariant under this action it follows that the action is carried over to an action of G on $\Omega^S \times_{x, \tau} A$. Now let P be the projection of $(\Omega^S \times_{x, \tau} A)^\wedge$ onto Ω . Then P is G -equivariant and it is open by Proposition 11. Thus (Ω, P) is an open σ -trivial regularization of the covariant system $(G, \Omega^S \times_{x, \tau} A)$, and we can apply Theorem 3 to see that $\Omega^S \times_{x, \tau} A$ is isomorphic to $\text{Ind}_{\Sigma^S}^G(\Sigma^S \times_{x_\Sigma, \tau_\Sigma} D)$. It is now a consequence of Proposition 10 that $\Omega^S \times_{x, \tau} A$ has continuous trace. ■

As a Corollary we get the following special case of Williams' theorem about transformation group algebras with continuous trace [32, Theorem 4.8].

COROLLARY 6. *Let Ω be a locally σ -trivial G -space such that S_ω is amenable for all $\omega \in \Omega$. Then the transformation group C^* -algebra $C^*(G, \Omega)$ has continuous trace if and only if $C^*(\Omega^S)$ has continuous trace.*

Clearly, it is in general not easy to decide whether a C^* -algebra constructed like $\Omega^S \times_{\alpha, \tau} A$ has continuous trace. A necessary condition is surely that $C^*(S_\omega, A_\omega, \tau_\omega)$ has continuous trace for all $\omega \in \Omega$, but the converse is not true even in the case where $A_\omega = \mathbb{C}$ for all $\omega \in \Omega$ and τ is trivial (see [5, Section 5]). But we now see that at least in some interesting special cases it is possible to reach better results.

For the remainder of this section let (G, A, τ) be a twisted covariant system such that A is of type I, \hat{A} is Hausdorff, and the action of G on $\Omega = \hat{A}$ is locally σ -trivial. It follows that A is isomorphic to a section algebra $\Gamma_0(E)$ of a C^* -bundle $p: E \rightarrow \Omega$ such that all fibers A_ω are elementary C^* -algebras; i.e., each A_ω is isomorphic to the algebra of compact operators on a Hilbert space \mathcal{H}_ω . We are now going to define locally unitary actions for twisted crossed products with continuously varying stabilizers, extending [24, Definition 5.1].

DEFINITION 6. Let A be a type I C^* -algebra with Hausdorff spectrum $\Omega = \hat{A}$, $H: \Omega \rightarrow \mathcal{K}(G)$ a continuous map, α an action of Ω^H on A , and τ a twisting map for α . Then α is *unitary relative to τ* if there exists a map $u: \Omega^H \rightarrow \prod_{\omega \in \Omega} \mathcal{U}(A_\omega)$ with the following properties:

- (1) $u(\omega, x) \in \mathcal{U}(A_\omega)$ and the maps $(\omega, x) \rightarrow u(\omega, x) a(\omega)$ and $(\omega, x) \rightarrow a(\omega) u(\omega, x)$ are continuous for all $a \in A$.
- (2) $u(\omega, \cdot)$ is a homomorphism from S_ω into $\mathcal{U}(A_\omega)$ for all $\omega \in \Omega$.
- (3) $\alpha_\omega(x)(a(\omega)) = u(\omega, x) a(\omega) u(\omega, x)^*$ for all $x \in S_\omega$.
- (4) $u(\omega, n) = \tau_\omega(n)$ for all $\omega \in \Omega$ and $n \in N_\tau$.

The map u is called a *unitary which implements α and τ* . α is called *locally unitary relative to τ* if for each $\omega_0 \in \Omega$ there exists an open neighborhood U of ω_0 such that α_U is unitary relative to τ_U . Furthermore, if (G, A, τ) is a twisted covariant system with continuous stabilizer map such that A is as above, then we say that the action of G is *unitary* or *locally unitary on the stabilizers relative to τ* if the action of Ω^S on A induced from the action of G is unitary or locally unitary relative to τ .

Remark 3. It is not hard to see that we can replace conditions (3) and (4) of the definition by the condition that $(\pi_\omega \circ u(\omega, \cdot), \pi_\omega)$ is a covariant representation of $(H_\omega, A_\omega, \tau_\omega)$. This shows that our definition coincides with the definition given in [19] in the special case of a trivial action of G on \hat{A} .

The assumption that an action is locally unitary is somewhat stronger than the assumption that all Mackey obstructions vanish or equivalently (and more appropriately in the non-separable case) that for each $\pi_\omega \in \hat{A}$ there exists a strictly continuous homomorphism $u: H_\omega \rightarrow \mathcal{U}(A_\omega)$ such

that $(\pi_\omega \circ u_\omega, \pi_\omega)$ is a covariant representation of $(H_\omega, A_\omega, \tau_\omega)$. Actions satisfying this property are called *pointwise unitary relative to τ* . If (G, A, τ) is a twisted covariant system such that A is of type I and Hausdorff, then we say that the action of G on A is *pointwise unitary on the stabilizers relative to τ* if each $\pi \in \hat{A}$ can be extended to a covariant representation of $(S_\omega, A_\omega, \tau_\omega)$ as described above. Due to a result of Rosenberg [29, Corollary 2.2] (see also [24, Proposition 5.5] and [19, Proposition 1.4]), it is known that in several interesting cases pointwise unitary actions are automatically locally unitary. We look at this more deeply at the end of this section.

Locally σ -trivial and locally unitary actions of abelian groups G on A have been studied extensively by Raeburn and Williams in [24] and the main result stated in [24, Section 6] is the fact that under these circumstances $C^*(G, A)^\wedge$ equipped with the dual action of \hat{G} is a σ -locally trivial \hat{G} -space. This implies in particular that $C^*(G, A)^\wedge$ is Hausdorff. Our main goal here is to generalize some of these results to actions of non-abelian groups and to answer a remaining question of whether $C^*(G, A)$ must have continuous trace if A has continuous trace (see [24, Remark 6.11]). As we can see the proof of our theorem will not be much complicated by allowing twisted covariant systems, so it seems to be reasonable to prove the results in the more general setting.

In the following $C^*(G, \Omega)$ denotes the transformation group algebra of the transformation group (G, Ω) which is identical to $C^*(G, C_0(\Omega))$. Note that if (G, A, τ) is a twisted covariant system such that $\Omega = \hat{A}$ is Hausdorff, then $\hat{G} = G/N_\tau$ acts canonically on Ω and the action of \hat{G} on Ω is (locally) σ -trivial if and only if the same is true for the action of G on Ω . We now state the main result of this section.

THEOREM 5. *Let (G, A, τ) be a twisted covariant system such that A is of type I and $\Omega = \hat{A}$ is Hausdorff. Suppose that Ω is a locally σ -trivial G -space and that the action of G on A is locally unitary on the stabilizers relative to τ . Furthermore let $\hat{G} = G/N_\tau$. Then the following is true:*

- (1) *$C^*(G, A, \tau)^\wedge$ is Hausdorff if and only if $C^*(\hat{G}, \Omega)^\wedge$ is Hausdorff.*
- (2) *$C^*(G, A, \tau)$ has continuous trace if and only if A and $C^*(\hat{G}, \Omega)$ have continuous trace.*

Before we start to prove this theorem we would like to state the following corollary which gives the answer to the problem stated in [24, Remark 6.11].

COROLLARY 7. *Let (G, A, τ) be as in Theorem 5 such that in addition (G, A, τ) satisfies one of the following conditions:*

- (1) S_ω/N_τ is abelian for all $\omega \in \Omega$.
- (2) G/N_τ is compact.
- (3) G/N_τ is a Lie group and the action of G/N_τ is proper; i.e., the map $P: \Omega \times G/N_\tau \rightarrow \Omega \times \Omega; (\omega, \dot{x}) \rightarrow (\omega, x\omega)$ is proper.

Then $C^*(G, A, \tau)$ has continuous trace if and only if A has continuous trace.

Proof. It is clear that $C^*(\Omega^\delta)$ has continuous trace if $\dot{S}_\omega = S_\omega/N_\tau$ is abelian for each $\omega \in \Omega$. Hence in this case the result follows from part (2) of the theorem and Corollary 6. If G/N_τ is compact, it was shown in [5] that $C^*(\Omega^\delta)$ has continuous trace and the same was shown in [5, Corollary 3] if G/N_τ is a Lie group which acts properly on Ω . ■

For the proof of the theorem we need the following lemma.

LEMMA 2. Let A be a type I C^* -algebra with Hausdorff spectrum Ω and $H: \Omega \rightarrow \mathcal{K}(G)$ a continuous map. Denote by $\Omega^H \times A$ the crossed product of Ω^H with A by the trivial action; i.e., α_ω is the trivial action of H_ω on A_ω for all $\omega \in \Omega$. Then $(\Omega^H \times A)^\wedge$ is homeomorphic to $C^*(\Omega^H)^\wedge$ and $\Omega^H \times A$ has continuous trace if and only if $C^*(\Omega^H)$ and A have continuous trace.

Proof. Let $C^*(\Omega^H) \hat{\otimes} A$ denote the unique tensor product of $C^*(\Omega^H)$ and A (A is nuclear by assumption), and define

$$I = \bigcap \{ \ker(\omega, \pi) \otimes \rho_\omega; \omega \in \Omega, \pi \in \hat{H}_\omega \} \subseteq C^*(\Omega^H) \hat{\otimes} A,$$

where ρ_ω denotes the irreducible representation of A related to ω . We claim that $\Omega^H \times A$ is isomorphic to $(C^*(\Omega^H) \hat{\otimes} A)/I$. In fact, if we define $\Phi: C_c(\Omega^H) \otimes A \rightarrow \Gamma_c(q^*E)$ by

$$(\Phi(f \otimes a))(\omega, x) = f(\omega, x) a(\omega),$$

then the image of Φ is dense in $\Omega^H \times A$ and it follows easily from the descriptions of the dual spaces of $C^*(\Omega^H) \hat{\otimes} A$ and $\Omega^H \times A$ that Φ can be extended to a $*$ -homomorphism onto $\Omega^H \times A$ with kernel I . This implies that $(\Omega^H \times A)^\wedge$ is homeomorphic to $C^*(\Omega^H)^\wedge$. Since the property of having continuous trace is inherited to tensor products and quotients it follows also that $\Omega^H \times A$ has continuous trace whenever $C^*(\Omega^H)$ and A have continuous trace.

Now suppose that $\Omega^H \times A$ has continuous trace. We define a $*$ -homomorphism Ψ from $\Gamma_c(q^*E)$ into A by

$$(\Psi f)(\omega) = \int_{H_\omega} f(\omega, x) d_{H_\omega} x.$$

Since the action of Ω^H on A is trivial we find that Ψ preserves multiplication and involution. Furthermore, if again ρ_ω denotes the representation corresponding to $\omega \in \Omega$, then

$$\rho_\omega(\Psi f) = (\omega, 1_{H_\omega}) \otimes \rho_\omega(f),$$

where 1_{H_ω} denotes the trivial representation of H_ω . Hence Ψ is norm-decreasing and extends to a $*$ -homomorphism from $\Omega^H \times A$ onto A . Thus A has continuous trace. To see that $C^*(\Omega^H)$ has continuous trace as well, let f and a be positive elements in the minimal dense ideals, the so-called Pedersen ideals, $m(C^*(\Omega^H))$ and $m(A)$ of $C^*(\Omega^H)$ and A , respectively. Then $f \otimes a$ is in the Pedersen ideal of $C^*(\Omega^H) \otimes A$. In this special case where A is nuclear this follows almost directly from the definition of the Pedersen ideal as given in [20, p. 175], but this is also true for tensor products of non-nuclear C^* -algebras (see [18, Lemma 8.9]). Since the Pedersen ideal is carried over by quotient maps it follows that $f \otimes a + I$ is in $m(\Omega^H \times A)$. Assume now that $(\omega, \pi) \rightarrow \text{tr}((\omega, \pi)(f))$ fails to be continuous at a point $(\omega_0, \pi_0) \in C^*(\Omega^H)^\wedge$. Choose $a \in m(A)$ such that $\text{tr}(\rho_{\omega_0}(a)) > 0$. Then it follows that the map $\Omega^H \times A \rightarrow \mathbb{C}$ given by

$$(\omega, \pi) \otimes \rho_\omega \rightarrow \text{tr}((\omega, \pi) \otimes \rho_\omega(f \otimes a)) = \text{tr}((\omega, \pi)(f)) \text{tr}(\rho_\omega(a))$$

is continuous. It is also clear that $\omega \rightarrow \text{tr}(\rho_\omega(a))$ is continuous, since we already know that A has continuous trace. Hence the positivity of $\text{tr}(\rho_{\omega_0}(a))$ contradicts our assumption that $(\omega, \pi) \rightarrow \text{tr}((\omega, \pi)(f))$ is not continuous at (ω_0, π_0) . The same argument shows that this map is finite on $C^*(\Omega^H)^\wedge$. Thus we conclude that $C^*(\Omega^H)$ has continuous trace. ■

Proof of Theorem 5. As in the proof of Theorem 4 we can localize the problem and it is enough to show that for each $\omega_0 \in \Omega$ there exists a G -invariant open neighborhood U of ω_0 such that the theorem is true for $C^*(G, \Gamma_0(E|U), \tau_U)$.

So let $\omega_0 \in \Omega$. Since Ω is a locally σ -trivial G -space we can find a G -invariant open neighborhood V_1 of ω_0 such that V_1 is a σ -trivial G -space. Let Σ_1 be a continuous section for V_1/G such that $\omega_0 \in \Sigma_1$. Furthermore, for each $\omega \in \Omega$ let α_ω be the action of S_ω on A_ω induced from the action of G on A . Since the action of G on A is locally unitary on the stabilizers relative to τ , there exists an open neighborhood $V_2 \subseteq V_1$ of ω_0 and a strictly continuous map $u: V_2^S \rightarrow \prod_{\omega \in V_2} \mathcal{U}(A)$ which implements α on V_2^S relative to τ . Now let $\Sigma = V_2 \cap \Sigma_1$ and $U = G(\Sigma) = \{x\omega; x \in G, \omega \in \Sigma\}$. Then U is an open G -invariant neighborhood of ω_0 and Σ is a section for U/G . Furthermore, the action of Σ^S on $\Gamma_0(E|U)$ is implemented by the restriction of the map u to Σ^S .

Hence, for the rest of the proof we can assume that Ω itself is σ -trivial, and that Σ is a continuous section for Ω/G . We may further assume that there is a strictly continuous map $u: \Sigma^S \rightarrow \prod_{\omega \in \Sigma} \mathcal{U}(A_\omega)$ which implements the action α_Σ of Σ^S on $D = F_0(E|\Sigma)$. Then it follows from Theorem 3 that $C^*(G, A, \tau)$ is Morita-equivalent to $\Sigma^S \times_{\alpha_\Sigma, \tau_\Sigma} D$. As in the lemma we denote by $\Sigma^S \times D$ the crossed product of Σ^S with D by the trivial action, where $\dot{S}: \Sigma \rightarrow \mathcal{K}(\dot{G})$ is given by $\dot{S}_\omega = S_\omega/N_\tau$, and let \dot{q} denote the projection from Σ^S onto Σ . Then the map $\Psi: \Gamma_c(q^*(E|\Sigma), \tau_\Sigma) \rightarrow \Gamma_c(\dot{q}^*(E|\Sigma))$ given by $(\Psi f)(\omega, x) = f(\omega, x)u(\omega, x)$ is easily seen to be a $*$ -homomorphism from a dense subalgebra of $\Sigma^S \times_{\alpha_\Sigma, \tau_\Sigma} D$ onto a dense subalgebra of $\Sigma^S \times D$. Since Ψ is isometric with respect to the norms $\|\cdot\|_1$, and $\|\cdot\|_*$, respectively, it extends to an isomorphism of $\Sigma^S \times_{\alpha_\Sigma, \tau_\Sigma} D$ onto $\Sigma^S \times D$. Hence $C^*(G, A, \tau)$ is Morita-equivalent to $\Sigma^S \times A$.

To prove (1) assume that $C^*(\dot{G}, \Omega)^\wedge$ is Hausdorff. Since $C^*(\dot{G}, \Omega)$ is Morita-equivalent to $C^*(\Sigma^S)$ by Theorem 3, it follows that $C^*(\Sigma^S)^\wedge$ is Hausdorff, too. $\Sigma^S \times A$ is then Hausdorff by Lemma 2, and the Morita equivalence between this algebra and $C^*(G, A, \tau)$ shows that $C^*(G, A, \tau)^\wedge$ is also Hausdorff. To prove the converse we use all these arguments in the opposite direction.

The Morita equivalence of $C^*(G, A, \tau)$ with $\Sigma^S \times D$ also implies [32, Theorem 2.15] that $C^*(G, A, \tau)$ has continuous trace if and only if $\Sigma^S \times D$ has continuous trace, which by Lemma 2 is equivalent to the assertion that both $C^*(\Sigma^S)$ and D have continuous trace. But this is equivalent to the conditions that $C^*(\dot{G}, \Omega)$ and A have continuous trace since $C^*(\dot{G}, \Omega)$ is Morita-equivalent to $C^*(\Sigma^S)$ and A is isomorphic to $\text{Ind}_{\Sigma^S}^G D$ (see Proposition 10 and Theorem 3). This finishes the proof. ■

We want to finish this section with some remarks about the assumptions made in Theorem 5. Let us first scrutinize the relations between pointwise unitary actions and locally unitary actions. Using Rosenberg's result [29, Corollary 2.2] it was shown by Raeburn and Williams [24, Proposition 5.5] that, in the case of separable covariant systems (G, A) such that G is compactly generated and A has continuous trace, pointwise unitary actions are always locally unitary, if in addition the stabilizer map is locally constant. We now extend the notion of locally constant stabilizer maps in order to extend this result to twisted covariant systems and actions of certain non-abelian groups.

DEFINITION 7. Let (G, A, τ) be a twisted covariant system with $\Omega = \hat{A}$ Hausdorff. Then the stabilizer map $S: \Omega \rightarrow \mathcal{K}(G)$ is called *locally constant relative to τ* if for each $\omega_0 \in \Omega$ there exists a neighborhood U of ω_0 and a continuous map $c: U \rightarrow \text{Aut}(G)$ with respect to the compact open topology on $\text{Aut}(G)$ such that

- (1) $c(\omega)(N_\tau) = N_\tau$ for all $\omega \in U$, and
- (2) there exists a closed subgroup H of G such that $c(\omega)(H) = S_\omega$ for all $\omega \in U$.

PROPOSITION 12. *Suppose that (G, A, τ) is a separable twisted covariant system such that A has continuous trace and the stabilizer map $S: \Omega \rightarrow \mathcal{K}(G)$, $\Omega = \hat{A}$, is locally constant. Suppose further that the action of G on A is pointwise unitary on the stabilizers relative to τ , and that each $\dot{S}_\omega = S_\omega/N_\tau$ satisfies the following conditions:*

- (1) *The second Moore-cohomology group $H^2(\dot{S}_\omega, \mathbb{T})$ is Hausdorff.*
- (2) *The abelianization $\dot{S}_\omega^{ab} = \dot{S}_\omega/[\dot{S}_\omega, \dot{S}_\omega]$ is compactly generated.*

Then G acts locally unitary on the stabilizers relative to τ .

Note that Conditions 1 and 2 are always satisfied if \dot{S}_ω is either compactly generated abelian or compact. This follows from [17, Theorem 7 and Proposition 6] and [16, Corollary 1]. But they are also true for several other classes of locally compact groups (see [29, Theorem 2.1]).

Proof of Proposition 12. Let $\omega_0 \in \Omega$ and U an open neighborhood of ω_0 such that there exist $c: U \rightarrow \text{Aut}(G)$ and $H \subseteq G$ as in Definition 7. We have $A = \Gamma_0(E)$ and define $B = \Gamma_0(E|U)$. Then we define an action of H on B and a twisting map $\tilde{\tau}: N_\tau \rightarrow \mathcal{U}(B)$ by

$${}^h b(\omega) = \alpha_\omega(c(\omega)(h))(b(\omega)), \quad h \in H, \quad b \in B, \quad \text{and} \quad \omega \in U,$$

and

$$(\tilde{\tau}(n)b)(\omega) = \tau_\omega(c(\omega)(n))b(\omega), \quad n \in N_\tau.$$

Then $(H, B, \tilde{\tau})$ is a twisted covariant system and it is clear that H acts pointwise or locally unitary on B relative to τ if and only if the action α_U of U^S on B is locally or pointwise unitary relative to τ , respectively. The Proposition follows now from [24, Proposition 1.4]. ■

Let us finally remark that the statements in Theorem 5 are in general not true if the actions are only assumed to be pointwise unitary. For instance [19, Example 2.4] shows that there are pointwise unitary actions of \mathbb{R} on a type I C^* -algebra A with Hausdorff dual space such that $C^*(\mathbb{R}, A)^\wedge$ is not Hausdorff. On the other hand, it is possible to construct a pointwise unitary action of \mathbb{R} such that A does not have continuous trace, but $C^*(\mathbb{R}, A)$ has continuous trace. For instance, let A be the transformation group algebra as given in the example of [13, p. 95] with the dual action of \mathbb{R} . It was shown in [13] that \hat{A} is Hausdorff but does not have

continuous trace. It is clear that the action of \mathbb{R} on A is pointwise unitary and it follows from Takai duality that $C^*(\mathbb{R}, A)$ has continuous trace.

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